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THE GLOBAL QUANTUM DUALITY PRINCIPLE

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Abstract. The "quantum duality principle" states that the quantization of a Lie bialgebra — via a quantum universal enveloping algebra (QUEA)— provides also a quantization of the dual Lie bialgebra (through its associated formal Poisson group) — via a quantum formal series Hopf algebra (QFSHA) — and, conversely, a QFSHA associated to a Lie bialgebra (via its associated formal Poisson group) yields a QUEA for the dual Lie bialgebra as well. Such a result was claimed true by Drinfeld (and proved by the author), and does hold in the framework of topological Hopf algebras, hence it is essentially "local" in nature. We give here a *global* formulation of this principle, dealing with standard Hopf algebras and with usual (i.e. non-formal) Poisson groups. The relevant examples of the special linear group, the Euclidean group and the Heisenberg group are studied in detail.

"Dualitas dualitatum
et omnia dualitas"
N. Barbecue, "Scholia"

Introduction

The quantum duality principle is known in literature with at least two formulations. One claims that quantum function algebras associated to dual Poisson groups can be taken to be dual — in the sense of duality of Hopf algebras — of each other; and similarly for quantum enveloping algebras (cf. [FRT] and [Se]). The second one, due to Drinfeld, states that any quantization of $F[[G]]$ "works" also — in a suitable sense — as a quantization of $U(\mathfrak{g}^*)$, and, conversely, any quantization of $U(\mathfrak{g})$ can be also "seen" as a quantization of $F[[G^*]]$: this is the point of view we are going to assume.

Our goal in this paper is twofold. First, in Section 2 we provide a *global* version of the quantum duality principle — dealing with quantizations built up over a ring of Laurent polynomials — which give information on the global data of our Poisson groups. This global formulation turns out to be especially useful in applications, for instance it allows

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one to formulate a nice "quantum duality principle for Poisson homogeneous spaces", cf. [CG]. Second, we illustrate this formulation of the principle by investigating in full detail three relevant examples: the semisimple groups (in Section 3), the Euclidean group (in Section 4), and the Heisenberg group (in Section 5).

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§ 1 Notation and terminology

1.1 The classical setting. Let k be a fixed field of characteristic zero. Let G be a connected algebraic group over k (or a Lie group, in the real or the p–adic case), and let $\mathfrak{g}$ be its tangent Lie algebra. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and by $F[G]$ the algebra of regular functions on G; moreover, let $F[[G]]$ be the (algebra of regular functions on the) formal group associated to G (or to $\mathfrak{g}$): this algebra can be seen either as the completion of $F[G]$ at the maximal ideal corresponding to the identity of G, or as $U(\mathfrak{g})^*$, the dual vector space of $U(\mathfrak{g})$. Both $U(\mathfrak{g})$ and $F[G]$ are Hopf algebras, whereas $F[[G]]$ is a *topological* Hopf algebra, for it has a coproduct which takes values in the topological tensor product (a suitable completion of the algebraic tensor product) of $F[[G]]$ with itself. Finally, there exist natural pairings of Hopf algebras (see §1.5 below) between $U(\mathfrak{g})$ and $F[G]$ and between $U(\mathfrak{g})$ and $F[[G]]$.

Now assume G is a Poisson group: then $\mathfrak{g}$ is a Lie bialgebra, $U(\mathfrak{g})$ is a co-Poisson Hopf algebra, $F[G]$ is a Poisson Hopf algebra, $F[[G]]$ is a topological Poisson Hopf algebra, and the Hopf pairings above are compatible with these additional co-Poisson and Poisson structures. Furthermore, the linear dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is a Lie bialgebra as well; then we can consider further objects G^*, $U(\mathfrak{g}^*)$, $F[G^*]$ and $F[[G^*]]$ as above.

1.2 The h–setting. Let V be a k–vector space; then the space $V[[h]]$ of formal power series in h with coefficients in V has a natural structure of a topological $k[[h]]$–module. We call a topological $k[[h]]$–module *topologically free* if it is isomorphic to $V[[h]]$ for some V. We denote by $\mathcal{A}$ the category of topologically free $k[[h]]$–modules, where morphisms are continuous $k[[h]]$–linear maps.

For any X in $\mathcal{A}$, we define the *dual* of X to be $X^\star := Hom_{\mathcal{A}}\big(X, k[[h]]\big)$, which again belongs to $\mathcal{A}$. Furthermore, we set $X_0 := X \big/ hX = k \otimes_{k[[h]]} X$: this is a k–module (via scalar extension $k[[h]] \rightarrow k$), which we call the *specialization* of X at $h = 0$; we shall also use such notation as $X \xrightarrow{h \rightarrow 0} \overline{Y}$ to mean that $X_0 \cong \overline{Y}$, and ${}_F X := k((h)) \otimes_{k[[h]]} X$.

A tensor structure in $\mathcal{A}$ is defined as follows: for X, Y in $\mathcal{A}$, define $X \widehat{\otimes} Y$ to be the projective limit of the $k[[h]]/h^n$–modules $\big(X/h^n X\big) \otimes_{k[[h]]/(h^n)} \big(Y/h^n Y\big)$ as $n \to \infty$. Notice then that $X \widehat{\otimes} Y$ is the h–adic completion of the algebraic tensor product $X \otimes_{k[[h]]} Y$.

Given X in $\mathcal{A}$, when saying that X is a *coalgebra*, resp. a *bialgebra*, resp. a *Hopf algebra*, we mean that X has such a structure with respect to the tensor structure in $\mathcal{A}$: in particular, X has a comultiplication which takes values in $X \widehat{\otimes} X$.

Definition 1.3. *("Local quantum groups" [or "algebras"]: cf. [Dr], § 7, [CP], § 6)*

(a) We call quantized universal enveloping algebra *(in short, QUEA) any Hopf algebra H (over $k[[h]]$) in $\mathcal{A}$ such that $H_0 := H/hH$ is isomorphic — as a Hopf algebra over k — to the universal enveloping algebra of a Lie algebra.*

(b) We call quantized formal series Hopf algebra *(in short, QFSHA) any Hopf algebra K (over $k[[h]]$) in $\mathcal{A}$ such that $K_0 := K/hK$ is isomorphic — as a topological Hopf algebra over k — to a Hopf algebra of formal series $k[[\{u_i \mid i \in J\}]]$ (for some set J).*

1.4 Remark. If H is a QUEA, then its specialization at $h = 0$, that is H_0, is a co-Poisson Hopf algebra; in particular, this means that, if $\mathfrak{g}$ is the Lie algebra such that $H_0 \cong U(\mathfrak{g})$, then $\mathfrak{g}$ is actually a *Lie bialgebra*; in this situation we shall write $H = U_h(\mathfrak{g})$. Similarly, if K is a QFSHA, then its specialization at $h = 0$ is a topological Poisson Hopf algebra; in particular, this means that K_0 is (the algebra of regular functions on) a formal Poisson group $F[[G]]$: in this situation we shall write $K = F_h[[G]]$ (cf. [Dr], § 7, or [CP], § 6, for details). By the way, we'd rather prefer such a terminology as "Quantum Formal Groups" instead of "Quantum Formal Series Hopf Algebras", but we stick to the latter in order to be consistent with Drinfeld use (which is nowadays standard).

Definition 1.5. *Let H, K be Hopf algebras (in any category) over a ring R. A pairing $\langle\ ,\ \rangle : H \times K \longrightarrow R$ is called* perfect *if it is non-degenerate; it is called a* Hopf pairing *if $\big\langle x, y_1 \cdot y_2 \big\rangle = \big\langle \Delta(x), y_1 \otimes y_2 \big\rangle$, $\big\langle x_1 \cdot x_2, y \big\rangle = \big\langle x_1 \otimes x_2, \Delta(y) \big\rangle$, $\langle x, 1 \rangle = \epsilon(x)$, $\langle 1, y \rangle = \epsilon(y)$, $\big\langle S(x), y \big\rangle = \big\langle x, S(y) \big\rangle$, for all $x, x_1, x_2 \in H$ and $y, y_1, y_2 \in K$.*

1.6 Drinfeld's functors. Let H be a Hopf algebra in $\mathcal{A}$. For every $n \in \mathbb{N}$, define $\Delta^n : H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$ (the counit), $\Delta^1 := id_H$, and $\Delta^n := \big(\Delta \otimes id_H^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ if $n > 2$. Then set $\delta_n = (id_H - \epsilon)^{\otimes n} \circ \Delta^n$ for all $n \in \mathbb{N}$. Finally, define

$$H' := \big\{\, a \in H \,\big|\, \delta_n(a) \in h^n H^{\otimes n} \,\big\} \quad (\subseteq H)$$

and endow H' with the induced topology.

Now let $I := Ker\big(H \xrightarrow{\ \epsilon\ } k[[h]] \longrightarrow k[[h]]/hk[[h] = k\big) = Ker\big(H \longrightarrow H/hH \xrightarrow{\ \bar{\epsilon}\ } k\big)$, a Hopf ideal of H, and consider the subalgebra $H^\times := \sum_{n \geq 0} h^{-n} I^n = \sum_{n \geq 0} \big(h^{-1} I\big)^n$ inside $k((h)) \otimes_{k[[h]]} H$: then define

$$H^\vee := h\text{–adic completion of } H^\times .$$

Theorem 1.7. *("The quantum duality principle"; cf. [Dr], §7, and [Ga4], §2)*

(a) If H is a QUEA, resp. a QFSHA, then H', resp. $H^\vee$, is a QFSHA, resp. a QUEA.

(b) The functors $H \longmapsto H'$ and $H \longmapsto H^\vee$ are inverse of each other.

(c) ("Quantum Duality Principle") Letting $\mathfrak{g}$, G, $\mathfrak{g}^*$ and G^* be as in § 1.1, we have

$$\big(U_h(\mathfrak{g})'\big)_0 = F[[G^*]] \, , \qquad \big(F_h[[G]]^\vee\big)_0 = U(\mathfrak{g}^*)$$

(notation of §1.4), that is to say $U_h(\mathfrak{g})' = F_h[[G^*]]$, $F_h[[G]]^\vee = U_h(\mathfrak{g}^*)$. $\square$

1.8 The q–setting. Let q be a formal variable, and consider a Hopf algebra H over $k(q)$: a subset $\overline{H}$ of H is called *a $k\big[q,q^{-1}\big]$–integer form* (or simply *a $k\big[q,q^{-1}\big]$–form*) *of* H if the following holds:

$$(a) \;\; \overline{H} \text{ is a Hopf subalgebra of } H \text{ over } k\big[q,q^{-1}\big] \, ; \qquad (b) \;\; k(q) \otimes_{k[q,q^{-1}]} \overline{H} = H \, .$$

For any $k\big[q,q^{-1}\big]$–module M, we set $M_1 := M\big/(q-1)M = k \otimes_{k[q,q^{-1}]} M$: this is a k–module (via scalar extension $k\big[q,q^{-1}\big] \to k$), which we call the *specialization* of M at $q=1$; we use also notation $M \xrightarrow{\,q \to 1\,} \overline{N}$ to mean that $M_1 \cong \overline{N}$.

Given two $k(q)$–modules A and B and a $k(q)$–bilinear pairing $A \times B \longrightarrow k(q)$, for any $k\big[q,q^{-1}\big]$–submodule $A_\bullet \subseteq A$ or $B_\bullet \subseteq B$ we set

$$A_\bullet^{\;\dagger} := \Big\{ b \in B \,\Big|\, \big\langle A_\bullet, b \big\rangle \subseteq k\big[q,q^{-1}\big] \Big\} \, , \quad B_\bullet^{\;\dagger} := \Big\{ a \in A \,\Big|\, \big\langle a, B_\bullet \big\rangle \subseteq k\big[q,q^{-1}\big] \Big\}$$

$$A_\bullet^{\;\perp_q} := \Big\{ b \in B \,\Big|\, \big\langle A_\bullet, b \big\rangle \subseteq (q-1) k\big[q,q^{-1}\big] \Big\} , \;\; B_\bullet^{\;\perp_q} := \Big\{ a \in A \,\Big|\, \big\langle a, B_\bullet \big\rangle \subseteq (q-1) k\big[q,q^{-1}\big] \Big\} .$$

Definition 1.9. *("Global quantum groups" [or "algebras"])*

(a) We call (global) quantized universal enveloping algebra *(in short, (G)QUEA) any pair $\big(U_q, \widehat{U}_q\big)$ such that U_q is a Hopf algebra over $k(q)$, $\widehat{U}_q$ is a $k\big[q,q^{-1}\big]$–integer form of U_q, and $\widehat{U}_1 := \big(\widehat{U}_q\big)_1$ is (isomorphic to) the universal enveloping algebra of a Lie algebra.*

(b) We call quantized function algebra *(in short, QFA) any pair $\big(F_q, \widehat{F}_q\big)$ such that F_q is a Hopf algebra over $k(q)$, $\widehat{F}_q$ is a $k\big[q,q^{-1}\big]$–integer form of F_q, and $\widehat{F}_1 := \big(\widehat{F}_q\big)_1$ is (isomorphic to) the algebra of regular functions of an algebraic group.*

Remark 1.10: as in §1.4, if $\big(U_q, \widehat{U}_q\big)$ is a QUEA, then $\widehat{U}_1$ is a co-Poisson Hopf algebra, so $\widehat{U}_1 \cong U(\mathfrak{g})$ where $\mathfrak{g}$ is a *Lie bialgebra* (the isomorphism then being one of *co-Poisson* Hopf algebras); in this situation we shall write $U_q = U_q(\mathfrak{g})$, $\widehat{U}_q = \widehat{U}_q(\mathfrak{g})$. Similarly, if $\big(F_q, \widehat{F}_q\big)$ is a QFA then $\widehat{F}_1$ is a Poisson Hopf algebra, thus $\widehat{F}_1 \cong F[G]$ where G is a *Poisson algebraic group* (the isomorphism then being one of *Poisson* Hopf algebras): in this situation we shall write $F_q = F_q[G]$, $\widehat{F}_q = \widehat{F}_q[G]$.

1.11 Drinfeld's-like functors. Let $\big(U_q, \widehat{U}_q\big)$ be a (G)QUEA (over $k(q)$), as in Definition 1.9. Consider linear maps δ_n, $n \in \mathbb{N}$, as in §1.6, and define

$$\widetilde{U}_q := \Big\{ a \in \widehat{U}_q \,\Big|\, \delta_n(a) \in (q-1)^n \widehat{U}_q^{\,\otimes n} \Big\} \quad \big(\subseteq \widehat{U}_q \big) \, .$$

Now let $\big(F_q, \widehat{F}_q\big)$ be a QFA (over $k(q)$), as in Definition 1.9. Let I be the kernel of the map $\widehat{F}_q \xrightarrow{\ \epsilon\ } k\big[q,q^{-1}\big] \xrightarrow{\ ev_1\ } k$, where ϵ is the counit of $\widehat{F}_q$ and ev_1 is the k–algebra morphism given by $ev_1(q) := 1$; again, this is the same as the kernel of the map $\widehat{F}_q \xrightarrow{\ ev_1\ } \widehat{F}_1 \xrightarrow{\ \bar{\epsilon}\ } k$ (where $\bar{\epsilon}$ is the counit of $\widehat{F}_1$), for the two maps coincide. Thus I is a maximal ideal of $\widehat{F}_q$. Then define

$$\widetilde{F}_q := \sum_{n \geq 0} (q-1)^{-n} I^n = \sum_{n \geq 0} \big((q-1)^{-1} I\big)^n \quad \big(\subseteq F_q\big).$$

1.12 Dual spaces. Let $\big(U_q, \widehat{U}_q\big)$ be a (G)QUEA: we define its linear *dual* to be the pair $\big({U_q}^*, {\widehat{U}_q}^{\,*}\big)$, whose entries are the (full) linear dual spaces ${U_q}^* := Hom_{k(q)}\big(U_q, k(q)\big)$ and ${\widehat{U}_q}^{\,*} := Hom_{k[q,q^{-1}]}\Big(\widehat{U}_q, k\big[q,q^{-1}\big]\Big)$. Both these spaces have natural structures of (topological) Hopf algebra (respectively over $k(q)$ and over $k\big[q,q^{-1}\big]$), such that the natural pairings $\langle\ ,\ \rangle : {U_q}^* \times U_q \longrightarrow k(q)$, $\langle\ ,\ \rangle : {\widehat{U}_q}^{\,*} \times \widehat{U}_q \longrightarrow k\big[q,q^{-1}\big]$ (given by evaluation) are perfect Hopf pairings.

For later use, we introduce also the Hopf algebra $\big({\widehat{U}_q}^{\,*}\big)^\vee$. Let $\widehat{I}_q^*$ be the kernel of the map ${\widehat{U}_q}^{\,*} \xrightarrow{\ \epsilon\ } k\big[q,q^{-1}\big] \xrightarrow{\ ev_1\ } k$ (a maximal ideal of ${\widehat{U}_q}^{\,*}$): then set

$$\big({\widehat{U}_q}^{\,*}\big)^\vee := \sum_{n \geq 0} (q-1)^{-n} \big(\widehat{I}_q^*\big)^n = \sum_{n \geq 0} \Big((q-1)^{-1} \widehat{I}_q^*\Big)^n \quad \big(\subseteq {U_q}^*\big).$$

§ 2 The global quantum duality principle

The present section is devoted to prove our main theoretical result, namely

Theorem 2.1. *("The global quantum duality principle")*

(a) If $\big(U_q, \widehat{U}_q\big)$ is a (G)QUEA, then $\big(U_q, \widetilde{U}_q\big)$ is a QFA; if $\big(F_q, \widehat{F}_q\big)$ is a QFA, then $\big(F_q, \widetilde{F}_q\big)$ is a (G)QUEA.

(b) The functors $\big(U_q, \widehat{U}_q\big) \mapsto \big(U_q, \widetilde{U}_q\big)$, $\big(F_q, \widehat{F}_q\big) \mapsto \big(F_q, \widetilde{F}_q\big)$, are inverse of each other.

(c) ("Global Quantum Duality Principle") If $\mathfrak{g}$, G, $\mathfrak{g}^*$ and G^* are as in § 1.1, then

$$\widetilde{U}_1(\mathfrak{g}) := \widetilde{U}_q(\mathfrak{g}) \Big/ (q-1)\, \widetilde{U}_q(\mathfrak{g}) \;=\; F[G^*]\,, \quad \widetilde{F}_1[G] := \widetilde{F}_q[G] \Big/ (q-1)\, \widetilde{F}_q[G] \;=\; U(\mathfrak{g}^*)$$

where the choice of the group G^* (among all the connected algebraic Poisson groups with tangent Lie bialgebra $\mathfrak{g}^*$) depends on the choice of the (G)QUEA $\big(U_q, \widehat{U}_q\big)$. In other words, $\big(U_q(\mathfrak{g}), \widetilde{U}_q(\mathfrak{g})\big)$ is a QFA for the Poisson group G^*, and $\big(F_q[G], \widetilde{F}_q[G]\big)$ is a (G)QFA for the Lie bialgebra $\mathfrak{g}^*$. $\square$

2.2 Localizing global quantum algebras. The present section is devoted to prove Theorem 2.1. Such a proof could be given by mimicking that of Theorem 1.7: one just has

to follow the pattern of proof provided in [Ga4], §2 (in fact, this will also add several further details to the statement we gave). Nevertheless, here we yield a proof which directly refers to the "local" quantum duality principle: namely, we get out of the "global" quantum algebras which are concerned, some "local" quantum algebras (as in §1) to which Theorem 1.7 can be applied, and then we formulate the result into "global terms".

Set, once and for all, $h := q-1$; then $q = 1+h$, and[1] there exists $q^{-1} = (1+h)^{-1} = \sum_{n=0}^{\infty} (-1)^n h^n \in k[[h]]$, so $k\big[q,q^{-1}\big] \subset k[[h]]$. Define

$$U_h(\mathfrak{g}) := h\text{–adic completion of } \ k[[h]] \otimes_{k[q,q^{-1}]} \widehat{U}_q(\mathfrak{g}) \, ;$$

then $U_h(\mathfrak{g}) \Big/ h\, U_h(\mathfrak{g}) = \widehat{U}_q(\mathfrak{g}) \Big/ (q-1)\, \widehat{U}_q(\mathfrak{g}) = U(\mathfrak{g})$, so $U_h(\mathfrak{g})$ is a QUEA — in the sense of Definition 1.3*(a)* — whence the QFSHA $U_h(\mathfrak{g})'$ is also defined, and definitions imply

$$\widetilde{U}_q(\mathfrak{g}) = \widehat{U}_q(\mathfrak{g}) \cap U_h(\mathfrak{g})' \, . \tag{2.1}$$

We proceed similarly with QFA's. Let $\big(F_q[G], \widehat{F}_q[G]\big)$ be a QFA, let I be the maximal ideal of $\widehat{F}_q[G]$ defined in §1.11: let $\widehat{F}_q[[G]]$ be the I–adic completion of $\widehat{F}_q[G]$, and define

$$F_h[[G]] := h\text{–adic completion of } \ k[[h]] \otimes_{k[q,q^{-1}]} \widehat{F}_q[[G]] \, ;$$

then $F_h[[G]] \Big/ h\, F_h[[G]] = \widehat{F}_q[[G]] \Big/ (q-1) \widehat{F}_q[[G]] =: \widehat{F}_1[[G]]$, and the latter equals the I_1–adic completion of $\widehat{F}_1 = F[G]$, that is just $F[[G]]$; so $F_h[[G]]$ is a QFSHA, hence the QUEA $F_h[[G]]^\vee$ is defined too. By the very definitions, we have

$$\widetilde{F}_q[G] = F_q[G] \cap F_h[[G]]^\vee \, . \tag{2.2}$$

Now let $\big(U_q(\mathfrak{g}), \widehat{U}_q(\mathfrak{g})\big)$ be a (G)QUEA, such that $\widehat{U}_q(\mathfrak{g})$ specializes to $U(\mathfrak{g})$ at $q=1$; then it follows that $\widehat{U}_q(\mathfrak{g})^*$ specializes to $U(\mathfrak{g})^* = F[[G]]$. Then define

$$\mathcal{U}_h(\mathfrak{g})^{\maltese} := \ h\text{–adic completion of } \ k[[h]] \otimes \widehat{U}_q(\mathfrak{g})^* \, .$$

It follows that $\mathcal{U}_h(\mathfrak{g})^{\maltese} \Big/ h\, \mathcal{U}_h(\mathfrak{g})^{\maltese} = \widehat{U}_q(\mathfrak{g})^* \Big/ (q-1)\, \widehat{U}_q(\mathfrak{g})^* = U(\mathfrak{g})^* = F[[G]]$, hence $\mathcal{U}_h(\mathfrak{g})^{\maltese}$ is a QFSHA, with $F[[G]]$ as semiclassical limit.

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[1]Usually, the link between h and q is set to be $q = \exp(h)$, hence $h = \log(q-1)$; but such a choice is equivalent to ours! In fact, both $\hat{q} := 1+h$ and $\tilde{q} := \exp(h)$ are invertible in $k[[h]]$, hence left multiplication by either of them yields a $k[[h]]$–module endomorphism of $k[[h]]$, which induces an endomorphism of any $k[[h]]$–module: denoting these by $\mu_{\hat{q}}$ and $\mu_{\tilde{q}}$ respectively, $\mu_{\hat{q}} \circ {\mu_{\tilde{q}}}^{-1}$ is an isomorphism from the induced $k\big[\tilde{q},\tilde{q}^{-1}\big]$–module structure to the induced $k\big[\hat{q},\hat{q}^{-1}\big]$–module structure of any $k[[h]]$–module. In addition, this isomorphism clearly commutes with specialization at $h=0$, hence yields also an isomorphism of the specialized modules, respectively at $\tilde{q}=1$ and at $\hat{q}=1$.

Proposition 2.3. *The natural Hopf pairing* $\langle\ ,\ \rangle : {U_q}^* \times U_q \longrightarrow k(q)$ *induces a perfect Hopf pairing* $\langle\ ,\ \rangle : \big(\widehat{U}_q{}^*\big)^\vee \times \widetilde{U}_q \longrightarrow k\big[q,q^{-1}\big]$, *whose specialization at* $q=1$ *is still perfect. More precisely, we have* $\widetilde{U}_q = \Big(\big(\widehat{U}_q{}^*\big)^\vee\Big)^\dagger$ *and* $\big(\widehat{U}_q{}^*\big)^\vee = {\widetilde{U}_q}{}^\dagger$.

Proof. By definition, any $\varphi \in \big(\widehat{U}_q{}^*\big)^\vee$ can be uniquely written as a (finite) sum $\varphi = \sum_{n=0}^N (q-1)^{-n} \varphi_n$ for some $N \in \mathbb{N}$ with $\varphi_n \in \big(\widehat{I}_q^*\big)^n$ for all $n = 0, 1, \dots, N$.

Now, the pairing $\langle\ ,\ \rangle : {U_q}^* \times U_q \longrightarrow k(q)$ clearly restricts to a similar pairing $\langle\ ,\ \rangle : \big(\widehat{U}_q{}^*\big)^\vee \times \widetilde{U}_q \longrightarrow k(q)$: we begin by proving that this takes values in $k\big[q,q^{-1}\big]$, that is $\Big\langle \big(\widehat{U}_q{}^*\big)^\vee, \widetilde{U}_q \Big\rangle \subseteq k\big[q,q^{-1}\big]$, which means that $\widetilde{U}_q \subseteq \Big(\big(\widehat{U}_q{}^*\big)^\vee\Big)^\dagger$ and $\big(\widehat{U}_q{}^*\big)^\vee \subseteq {\widetilde{U}_q}{}^\dagger$.

First, take $c_1, \dots, c_n \in \widehat{I}_q^*$; then $\langle c_i, 1 \rangle = \epsilon(c_i)$, so $\langle c_i, 1 \rangle \in (q-1) \cdot k\big[q,q^{-1}\big]$, because $c_i \in \widehat{I}_q^* := Ker\Big(\widehat{U}_q{}^* \xrightarrow{\ \epsilon\ } k\big[q,q^{-1}\big] \xrightarrow{\ ev_1\ } k\Big)$ ($i = 1, \dots, n$). Given $y \in \widetilde{U}_q$, consider

$$\left\langle \prod_{i=1}^n c_i \,, y \right\rangle = \Big\langle \otimes_{i=1}^n c_i \,, \Delta^{(n)}(y) \Big\rangle = \left\langle \otimes_{i=1}^n c_i \,, \sum_{\Psi \subseteq \{1,\dots,n\}} \delta_\Psi(y) \right\rangle = \sum_{\Psi \subseteq \{1,\dots,n\}} \big\langle \otimes_{i=1}^n c_i \,, \delta_\Psi(y) \big\rangle ;$$

here we use the following formulae relating the Δ^n's and the δ_m's in any Hopf algebra H:

$$\delta_\Sigma = \sum_{\Psi \subset \Sigma} (-1)^{n-|\Psi|} \Delta_\Psi \ , \quad \delta_\emptyset := \epsilon \ , \qquad \Delta_\Sigma = \sum_{\Psi \subseteq \Sigma} \delta_\Psi \ , \quad \Delta_\emptyset := \Delta^0$$

where for any ordered subset $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, we define the morphism $j_\Sigma \colon H^{\otimes k} \longrightarrow H^{\otimes n}$ by $j_\Sigma(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_{i_m} := a_m$ for $1 \leq m \leq k$, and we set $\Delta_\Sigma := j_\Sigma \circ \Delta^k$, $\delta_\Sigma := j_\Sigma \circ \delta_k$.

Now look at the generic summand in the last term in the formula above. Let $|\Psi| = t$ ($t \leq n$): then $\big\langle \otimes_{i=1}^n c_i, \delta_\Psi(y) \big\rangle = \big\langle \otimes_{i \in \Psi} c_i, \delta_t(y) \big\rangle \cdot \prod_{j \not\in \Psi} \langle c_j, 1 \rangle$, by definition of δ_Ψ. Thanks to the previous analysis, we have $\prod_{j \not\in \Psi} \langle c_j, 1 \rangle \in (q-1)^{n-t} k\big[q,q^{-1}\big]$ (perhaps zero!), and $\big\langle \otimes_{i \in \Psi} c_i, \delta_t(y) \big\rangle \in (q-1)^t k\big[q,q^{-1}\big]$ because $y \in \widetilde{U}_q$; thus $\big\langle \otimes_{i \in \Psi} c_i, \delta_t(y) \big\rangle \cdot \prod_{j \not\in \Psi} \langle c_j, 1 \rangle \in (q-1)^n k\big[q,q^{-1}\big]$, whence $\big\langle \prod_{i=1}^n c_i, y \big\rangle \in (q-1)^n k\big[q,q^{-1}\big]$. We conclude that

$$\langle \psi, y \rangle \in (q-1)^n k\big[q,q^{-1}\big] \qquad \forall \ \ \psi \in \big(\widehat{I}_q^*\big)^n, \ \ y \in \widetilde{U}_q \, . \tag{2.3}$$

Now take $\varphi = \sum_{n=0}^N (q-1)^{-n} \varphi_n \in \big(\widehat{U}_q{}^*\big)^\vee$ and $x \in \widetilde{U}_q$. Since $\varphi_n \in \big(\widehat{I}_q^*\big)^n$ for all n, we have (thanks to (2.3))

$$\langle \varphi, x \rangle = \sum_{n=0}^N (q-1)^{-n} \langle \varphi_n, x \rangle \in \sum_{n=0}^N (q-1)^{-n} (q-1)^n k\big[q,q^{-1}\big] = k\big[q,q^{-1}\big] \ , \quad \text{q.e.d.}$$

Thus the restriction of the pairing $\langle\ ,\ \rangle \colon {U_q}^* \times U_q \longrightarrow k(q)$ to $\big(\widehat{U}_q{}^*\big)^\vee \times \widetilde{U}_q$ does take values in $k\big[q,q^{-1}\big]$, as expected; therefore $\widetilde{U}_q \subseteq \Big(\big(\widehat{U}_q{}^*\big)^\vee\Big)^\dagger$ and $\big(\widehat{U}_q{}^*\big)^\vee \subseteq {\widetilde{U}_q}{}^\dagger$.

Second, we prove that $\Big(\big({\widehat{U}_q}^* \big)^\vee \Big)^\dagger \subseteq \widetilde{U}_q$; to this end, we revert the previous argument.

Let $\psi \in \Big(\big({\widehat{U}_q}^* \big)^\vee \Big)^\dagger$: then $\Big\langle (q-1)^{-s} \big(\widehat{I}_q^*\big)^s, \psi \Big\rangle \in k\big[q,q^{-1}\big]$ hence $\Big\langle \big(\widehat{I}_q^*\big)^s, \psi \Big\rangle \in (q-1)^s \cdot k\big[q,q^{-1}\big]$, for all s. For $s=0$ this gives $\big\langle {\widehat{U}_q}^*, \psi \big\rangle \in k\big[q,q^{-1}\big]$, whence $\psi \in \widehat{U}_q$.

Now let $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \widehat{I}_q^*$; then

$$\Big\langle \otimes_{k=1}^n i_k \, , \delta_n(\psi) \Big\rangle = \left\langle \otimes_{k=1}^n i_k \, , \sum_{\Psi \subseteq \{1,\dots,n\}} (-1)^{n-|\Psi|} \cdot \Delta_\Psi(\psi) \right\rangle =$$

$$= \sum_{\Psi \subseteq \{1,\dots,n\}} (-1)^{n-|\Psi|} \cdot \left\langle \prod_{k \in \Psi}^n i_k \, , \psi \right\rangle \cdot \prod_{k \notin \Psi} \epsilon(i_k) \in \sum_{\Psi \subseteq \{1,\dots,n\}} \left\langle \big(\widehat{I}_q^*\big)^{|\Psi|}, \psi \right\rangle \cdot (q-1)^{n-|\Psi|} \, k\big[q,q^{-1}\big] \subseteq$$

$$\subseteq \sum_{s=0}^n (q-1)^s \cdot (q-1)^{n-s} \, k\big[q,q^{-1}\big] = (q-1)^n \, k\big[q,q^{-1}\big]$$

therefore $\Big\langle \big(\widehat{I}_q^*\big)^{\otimes n}, \delta_n(\psi) \Big\rangle \subseteq (q-1)^n k\big[q,q^{-1}\big]$. In addition, ${\widehat{U}_q}^*$ splits as ${\widehat{U}_q}^* = k \cdot 1_{{\widehat{U}_q}^*} \oplus Ker\big(ev_1 \circ \epsilon_{{\widehat{U}_q}^*}\big) = k \cdot 1_{{\widehat{U}_q}^*} \oplus \widehat{I}_q^*$, so $\big({\widehat{U}_q}^*\big)^{\otimes n}$ splits into the direct sum of $\big(\widehat{I}_q^*\big)^{\otimes n}$ plus other direct summands which are tensor products on their own in which at least one tensor factor is $k \cdot 1_{{\widehat{U}_q}^*}$. Since $J_{\widehat{U}_q} := Ker(\epsilon_{\widehat{U}_q}) = \big\{ y \in \widehat{U}_q \,\big|\, \langle 1_{{\widehat{U}_q}^*}, y \rangle = 0 \big\} = \big\langle k \cdot 1_{{\widehat{U}_q}^*} \big\rangle^\perp$ (the orthogonal space of $\big\langle k \cdot 1_{{\widehat{U}_q}^*} \big\rangle$), we have $\Big\langle \big({\widehat{U}_q}^*\big)^{\otimes n}, j \Big\rangle = \Big\langle \big(\widehat{I}_q^*\big)^{\otimes n}, j \Big\rangle$ for any $j \in {J_{\widehat{U}_q}}^{\otimes n}$. Now observe that $\delta_n(\psi) \in {J_{\widehat{U}_q}}^{\otimes n}$, by the very definition of δ_n ; this and the previous analysis together give $\Big\langle \big({\widehat{U}_q}^*\big)^{\otimes n}, \delta_n(\psi) \Big\rangle \subseteq (q-1)^n \, k\big[q,q^{-1}\big]$, whence we get $\delta_n(\psi) \in (q-1)^n \widehat{U}_q^{\otimes n}$, q.e.d.

Third, we prove that ${\widetilde{U}_q}^\dagger = \big({\widehat{U}_q}^*\big)^\vee$. We already know that ${\widetilde{U}_q}^\dagger \supseteq \big({\widehat{U}_q}^*\big)^\vee$; so assume there exists $\phi \in {\widetilde{U}_q}^\dagger \setminus \big({\widehat{U}_q}^*\big)^\vee$: then we have $\widetilde{U}_q \subseteq \Big(\big({\widehat{U}_q}^*\big)^\vee \oplus k\big[q,q^{-1}\big] \cdot \phi \Big)^\dagger$, which yields a contradiction with $\widetilde{U}_q = \Big(\big({\widehat{U}_q}^* \big)^\vee \Big)^\dagger$ because $\big({\widehat{U}_q}^*\big)^\vee \subsetneqq \big({\widehat{U}_q}^*\big)^\vee \oplus k\big[q,q^{-1}\big] \cdot \phi$ implies instead $\Big(\big({\widehat{U}_q}^* \big)^\vee \Big)^\dagger \supsetneqq \Big(\big({\widehat{U}_q}^*\big)^\vee \oplus k\big[q,q^{-1}\big] \cdot \phi \Big)^\dagger$. Thus we conclude that ${\widetilde{U}_q}^\dagger = \big({\widehat{U}_q}^*\big)^\vee$, q.e.d.

Fourth, as the pairing $\langle \ , \ \rangle \colon \big({\widehat{U}_q}^*\big)^\vee \times \widetilde{U}_q \longrightarrow k\big[q,q^{-1}\big]$ of course is still a *Hopf* pairing, we are only have to prove that it is *perfect* as well. This pairing induces a natural morphism of Hopf algebras $\big({\widehat{U}_q}^*\big)^\vee \xrightarrow{\ \mu\ } \big(\widetilde{U}_q\big)^* := Hom\Big(\widetilde{U}_q, k\big[q,q^{-1}\big]\Big)$, $f \mapsto \big\langle f, - \big\rangle$. The non-degeneracy of the pairing corresponds, by definition, to the triviality of both the right and left kernel of the pairing: the first condition is trivially verified, and the second is equivalent to the injectivity of μ ; hence our task is to prove that $Ker(\mu) = 0$.

Let $s \in \big({\widehat{U}_q}^*\big)^\vee := \sum_{n \geq 0} (q-1)^{-n} \big(\widehat{I}_q^*\big)^n$, $s \neq 0$; then $s = (q-1)^{-N} \cdot s_+$, with $s_+ = \sum_{k=0}^N s_{N-k} (q-1)^k \in {\widehat{U}_q}^*$, $s_+ \neq 0$, and $\mu(s) = 0 \iff \mu(s_+) = 0$. Therefore, for the injectivity of μ to be proved it is enough for us to show that $\mu(s_+) \neq 0$.

Since $s_+ \in {\widehat{U}_q}^* \subseteq {U_q}^*$ and the latter vector space is in perfect pairing with U_q , there exist $\kappa \in U_q$ and a vector subspace $V \leq {U_q}^*$ (the orthogonal of $k(q) \cdot \kappa$) such that

${U_q}^* = k(q) \cdot s_+ \oplus V$ and $\langle s_+, \kappa \rangle \neq 0$, $\langle V, \kappa \rangle = 0$; moreover, there exists $c \in k\big[q, q^{-1}\big] \setminus \{0\}$ such that $\langle s_+, c \cdot \kappa \rangle = c \cdot \langle s_+, \kappa \rangle \in k\big[q, q^{-1}\big]$, so that $\kappa_+ := c \cdot \kappa \in \Big(\big({\widehat{U}_q}^* \big)^\vee \Big)^\dagger$. But we have already proved the equality $\Big(\big({\widehat{U}_q}^* \big)^\vee \Big)^\dagger = \widetilde{U}_q$, hence $\kappa_+ \in \widetilde{U}_q$, and $\langle s_+, \kappa_+ \rangle \neq 0$: therefore $\mu(s_+) \neq 0$, q.e.d.

Finally, proving that the specialization of the pairing $\langle \ , \ \rangle : \big({\widehat{U}_q}^* \big)^\vee \times \widetilde{U}_q \longrightarrow k\big[q, q^{-1}\big]$ (at $q = 1$) is still perfect amounts to show that $\Big(\big({\widehat{U}_q}^* \big)^\vee \Big)^{\perp_q} = (q-1)\, \widetilde{U}_q$ and ${\widetilde{U}_q}^{\perp_q} = (q-1) \big({\widehat{U}_q}^* \big)^\vee$. Now, let $u \in \Big(\big({\widehat{U}_q}^* \big)^\vee \Big)^{\perp_q}$: then $u_- := (q-1)^{-1} u \in \Big(\big({\widehat{U}_q}^* \big)^\vee \Big)^\dagger = \widetilde{U}_q$, hence $u = (q-1)\, u_- \in (q-1)\, \widetilde{U}_q$, q.e.d. And similarly for the other case. $\square$

Proposition 2.4. *Let* $\big(U_q(\mathfrak{g}), \widehat{U}_q(\mathfrak{g}) \big)$ *be a (G)QUEA. Then* $\widetilde{U}_q(\mathfrak{g})$ *is a* $k\big[q, q^{-1}\big]$*–integer form of* $U_q(\mathfrak{g})$*.*

Proof. Consider $U_h(\mathfrak{g})$ and $U_h(\mathfrak{g})'$. By (2.1), we have that $\widetilde{U}_q(\mathfrak{g})$ is a Hopf subalgebra (of $U_q(\mathfrak{g})$ over $k\big[q, q^{-1}\big]$) because $U_h(\mathfrak{g})'$ is a (topological) Hopf subalgebra (of $U_h(\mathfrak{g})$), thanks to Theorem 1.7*(a)*. Thus we only have to show that $U_q(\mathfrak{g}) = k(q) \otimes_{k[q,q^{-1}]} \widetilde{U}_q(\mathfrak{g})$.

Definitions give $\widetilde{U}_q(\mathfrak{g}) \subseteq \widehat{U}_q(\mathfrak{g})$, whence $k(q) \otimes_{k[q,q^{-1}]} \widetilde{U}_q(\mathfrak{g}) \subseteq k(q) \otimes_{k[q,q^{-1}]} \widehat{U}_q(\mathfrak{g}) = U_q(\mathfrak{g})$; moreover, Proposition 2.3 implies that $k(q) \otimes_{k[q,q^{-1}]} \widetilde{U}_q(\mathfrak{g})$ is perfectly paired (over $k(q)$) with $k(q) \otimes_{k[q,q^{-1}]} \big({\widehat{U}_q}^* \big)^\vee$, while $k(q) \otimes_{k[q,q^{-1}]} \widehat{U}_q(\mathfrak{g}) = U_q(\mathfrak{g})$ is perfectly paired (over $k(q)$) with $k(q) \otimes_{k[q,q^{-1}]} {\widehat{U}_q}^* \subseteq U_q(\mathfrak{g})^*$. Now suppose $k(q) \otimes_{k[q,q^{-1}]} \widetilde{U}_q(\mathfrak{g}) \subsetneqq U_q(\mathfrak{g})$: then there exists $\phi \in U_q(\mathfrak{g})^*$ such that $\phi \neq 0$ and $\phi \Big(k(q) \otimes_{k[q,q^{-1}]} \widetilde{U}_q(\mathfrak{g}) \Big) = 0$; in addition, we can find such a ϕ inside the space $k(q) \otimes_{k[q,q^{-1}]} {\widehat{U}_q}^*$: but $k(q) \otimes_{k[q,q^{-1}]} {\widehat{U}_q}^* = k(q) \otimes_{k[q,q^{-1}]} \big({\widehat{U}_q}^* \big)^\vee$ (by definitions), so we end up with a contradiction. $\square$

Proposition 2.5. $\widetilde{F}_q[G]$ *is a* $k\big[q, q^{-1}\big]$*–integer form of* $F_q[G]$*.*

Proof. Let $F_h[[G]]$ be the QFSHA defined in §2.2, and consider the QUEA $F_h[[G]]^\vee$. By (2.2), we have that $\widetilde{F}_q[G]$ is a Hopf subalgebra (of $F_q[G]$ over $k\big[q, q^{-1}\big]$) because $F_h[[G]]^\vee$ is a (topological) Hopf algebra, thanks to Theorem 1.7*(a)*. Moreover, one clearly has $F_q[G] = k(q) \otimes_{k[q,q^{-1}]} \widetilde{F}_q[G]$ because $F_q[G] = k(q) \otimes_{k[q,q^{-1}]} \widehat{F}_q[G]$ (by hypothesis), and $\widehat{F}_q[G] \subseteq \widetilde{F}_q[G]$, by construction. The claim follows. $\square$

Proposition 2.6.

$$\widetilde{U}_q(\mathfrak{g}) \Big/ (q-1)\, \widetilde{U}_q(\mathfrak{g}) \, = \, F[G^*]$$

for some connected algebraic Poisson group G^* *whose tangent Lie bialgebra is* $\mathfrak{g}^*$ *.*

Proof. Consider the QFSHA $\mathcal{U}_h(\mathfrak{g})^{\maltese}$ defined in §2.2, whose semiclassical limit is $F[[G]]$; then its associated QUEA $\big(\mathcal{U}_h(\mathfrak{g})^{\maltese} \big)^\vee$ has semiclassical limit $U(\mathfrak{g}^*)$, due to Theorem 1.7*(c)*.

Let $\widehat{I} := Ker\Big(\widehat{U}_q(\mathfrak{g})^* \xrightarrow{\ \epsilon\ } k\big[q,q^{-1}\big] \longrightarrow k\Big)$ and $\widehat{\mathcal{I}} := Ker\Big(\mathcal{U}_h(\mathfrak{g})^{\circledast} \xrightarrow{\ \epsilon\ } k[[h]] \longrightarrow k\Big)$; then definitions imply that $\widehat{\mathcal{I}}$ is the h–adic closure (completion) of $\widehat{I}$, so $\widehat{\mathcal{I}} = \coprod_{n=0}^{\infty} h^n \cdot \widehat{I}$ (hereafter, the notation $\coprod_{n=0}^{\infty} T_n$ is used to denote the completion of the space $\sum_{n=0}^{\infty} T_n$ with respect to the filtration afforded by the family of subspaces $\big\{ \sum_{k=0}^{N} T_k \,\big|\, N \in \mathbb{N} \big\}$). As $\Big(\mathcal{U}_h(\mathfrak{g})^{\circledast}\Big)^{\vee}$ is the h–adic completion of the space $S := \sum_{n=0}^{\infty} \Big(h^{-1} \cdot \widehat{\mathcal{I}}\Big)^n$, its limit at $h=0$ is the same as that of S: for the latter, we have

$$S := \sum_{n=0}^{\infty} \Big(h^{-1} \cdot \widehat{\mathcal{I}}\Big)^n = \sum_{n=0}^{\infty} \left(h^{-1} \cdot \Big(\coprod_{k=0}^{\infty} h^k \cdot \widehat{I}\Big)\right)^n = \sum_{n=0}^{\infty} \left(h^{-1} \cdot \widehat{I} + h \cdot h^{-1} \cdot \coprod_{s=0}^{\infty} h^s \widehat{I}\right)^n =$$
$$= \sum_{n=0}^{\infty} \Big(h^{-1} \cdot \widehat{I} + h \cdot h^{-1} \cdot \widehat{\mathcal{I}}\Big)^n \subseteq \sum_{r=0}^{\infty} \Big(h^{-1} \cdot \widehat{I}\Big)^r + h \cdot \sum_{t=0}^{\infty} \Big(h^{-1} \cdot \widehat{\mathcal{I}}\Big)^t =$$
$$= \sum_{k=0}^{\infty} \Big((q-1)^{-1} \cdot \widehat{I}\Big)^k + h \cdot S = \Big(\widehat{U}_q(\mathfrak{g})^*\Big)^{\vee} + h \cdot S$$

for $\Big(\widehat{U}_q(\mathfrak{g})^*\Big)^{\vee} := \sum_{n=0}^{\infty} \Big((q-1)^{-1} \cdot \widehat{I}\Big)^n$. Therefore $\Big(\widehat{U}_q(\mathfrak{g})^*\Big)^{\vee}$ has the same semiclassical limit as S, hence the same of $\Big(\mathcal{U}_h(\mathfrak{g})^{\circledast}\Big)^{\vee}$, that is $U(\mathfrak{g}^*)$: thus $\Big(\widehat{U}_q(\mathfrak{g})^*\Big)^{\vee} \xrightarrow{\ q \to 1\ } U(\mathfrak{g}^*)$.

Now, $U_h(\mathfrak{g})$ has semiclassical limit $U(\mathfrak{g})$, hence the associated QFSHA $U_h(\mathfrak{g})'$ has semiclassical limit $F[[G^*]]$ (by Theorem 1.7*(c)*). By the very definitions, $\widetilde{U}_q(\mathfrak{g})$ is a Hopf subalgebra of $U_h(\mathfrak{g})'$, so its semiclassical limit $\widetilde{U}_1(\mathfrak{g}) := \widetilde{U}_q(\mathfrak{g}) \Big/ (q-1)\, \widetilde{U}_q(\mathfrak{g})$ is a (*standard*, i.e. non-topological) (Poisson) Hopf subalgebra of $F[[G^*]]$. This means that the maximal spectrum of $\widetilde{U}_1(\mathfrak{g})$ is an algebraic group $\varGamma$, with $\widetilde{U}_1(\mathfrak{g}) \cong F[\varGamma]$; so $F[\varGamma]$ is a Poisson Hopf subalgebra of $F[[G^*]]$, which means that $\varGamma$ is a Poisson algebraic subgroup of a suitable Poisson algebraic group G^* (whose tangent Lie bialgebra is $\mathfrak{g}^*$). Now, by Proposition 2.3 we have a perfect Hopf pairing $\langle\ ,\ \rangle : \Big(\widehat{U}_q(\mathfrak{g})^*\Big)^{\vee} \times \widetilde{U}_q(\mathfrak{g}) \longrightarrow k\big[q,q^{-1}\big]$ which for $q \to 1$ yields a similar pairing between the specializations of $\Big(\widehat{U}_q(\mathfrak{g})^*\Big)^{\vee}$ and of $\widetilde{U}_q(\mathfrak{g})$, i.e. a perfect Hopf pairing $\langle\ ,\ \rangle : U(\mathfrak{g}^*) \times F[\varGamma] \longrightarrow k$ which is nothing but the restriction of the natural evaluation pairing between $U(\mathfrak{g}^*)$ and $F[[G^*]]$. At last, the non-degeneracy of the last pairing implies that $\varGamma$ is connected — because the *right* kernel ($\subseteq F[\varGamma]$) of the pairing is trivial — and $\varGamma = G^*$ (that is to say, $\mathrm{Lie}(\varGamma) = \mathfrak{g}^*$) — because the *left* kernel ($\subseteq U\big(\mathfrak{g}^*\big)$) is trivial. The claim follows. □

Proposition 2.7.

$$\widetilde{F}_q[G] \Big/ (q-1)\, \widetilde{F}_q[G] \;=\; U\left(\mathfrak{g}^*\right) \, .$$

Proof. Let I be the maximal ideal of $\widehat{F}_q[G]$ defined in §2.2, let $\widehat{F}_q[[G]]$ be the I–adic completion of $\widehat{F}_q[G]$, and let I_∞ be the closure of I in $\widehat{F}_q[[G]]$, which is $I_\infty = \coprod_{k=1}^{+\infty} I^k$. Finally, define $\widehat{F}_q[[G]]^{\vee} := \sum_{n=0}^{+\infty} (q-1)^{-n} I_\infty^n = \sum_{n=0}^{+\infty} \big((q-1)^{-1} I_\infty\big)^n \ \Big(\subset k(q) \otimes_{k[q,q^{-1}]} \widehat{F}_q[[G]]\Big)$.

Now consider $F_h[[G]]$ defined in §2.2, and the QUEA $F_h[[G]]^\vee$ associated to it. Then $F_h[[G]]^\vee$ is nothing but the $(q-1)$–adic completion of $\widehat{F}_q[[G]]^\vee$: therefore

$$\widehat{F}_q[[G]]^\vee \Big/ (q-1)\widehat{F}_q[[G]]^\vee \; = \; F_h[[G]]^\vee \Big/ hF_h[[G]]^\vee \; = \; U(\mathfrak{g}^*) \tag{2.4}$$

where the second equality holds by Theorem 1.7*(c)*. Furthermore, we have also $\widehat{F}_q[[G]]^\vee :=$
$= \sum_{n=0}^{\infty} \Big((q-1)^{-1} I_\infty \Big)^n = \sum_{n=0}^{\infty} \Big((q-1)^{-1} \coprod_{k=1}^{\infty} I^k \Big)^n = \sum_{n=0}^{\infty} \Big(\coprod_{k=1}^{\infty} (q-1)^{k-1} \Big((q-1)^{-1} I \Big)^k \Big)^n \subseteq$
$\subseteq \sum_{n=0}^{\infty} \big((q-1)^{-1} I \big)^n + (q-1) \coprod_{k=0}^{\infty} \Big((q-1)^{-1} \cdot I \Big)^k \subseteq \widetilde{F}_q[G] + (q-1)\widehat{F}_q[[G]]^\vee$, therefore $\widehat{F}_q[[G]]^\vee \Big/ (q-1)\widehat{F}_q[[G]]^\vee = \widetilde{F}_q[G] \Big/ (q-1)\widetilde{F}_q[G]$, and this and (2.4) give the claim. $\square$

Corollary (of the proof) 2.8. *The pair* $\Big(k(q) \otimes_{k[q,q^{-1}]} \Big(\widehat{U}_q(\mathfrak{g})^* \Big)^\vee, \Big(\widehat{U}_q(\mathfrak{g})^* \Big)^\vee \Big)$ *is a (G)QUEA, and the semiclassical limit of* $\Big(\widehat{U}_q(\mathfrak{g})^* \Big)^\vee$ *is* $U(\mathfrak{g}^*)$. $\square$

Proof of Theorem 2.1. Parts *(a)* and *(c)* are proved by the whole of Propositions 2.4-7. For part *(b)*, we proceed in steps: $\big(U_q, \widehat{U}_q\big)$ will be a (G)QUEA and $\big(F_q, \widehat{F}_q\big)$ a QFA.

<u>*Claim 1:*</u> *(a)* $\widehat{F}_q \subseteq \widetilde{\widetilde{F}}_q$ *and* *(b)* $\widehat{U}_q \supseteq \widetilde{\widetilde{U}}_q$.

Proof of Claim 1. From $\delta_n = (\mathrm{id}_H - \epsilon)^{\otimes n} \circ \Delta^n$ (for any $n \in \mathbb{N}$, and H any Hopf algebra) we have that $\delta_n(H) \subseteq {J_H}^{\otimes n}$, where $J_H := Ker(\epsilon_H)$; for $H = \widehat{F}_q$ this yields $\delta_n\big(\widehat{F}_q\big) \subseteq {J_{\widehat{F}_q}}^{\otimes n} = (q-1)^n \big((q-1)^{-1} J_{\widehat{F}_q} \big)^{\otimes n} \subseteq (q-1)^n \, {\widetilde{F}_q}^{\otimes n}$, which gives $\widehat{F}_q \subseteq \widetilde{\widetilde{F}}_q$, so part *(a)* is done. As for part *(b)*, let $I' := Ker\Big(\widetilde{U}_q \xrightarrow{\;\epsilon\;} k\big[q,q^{-1}\big] \xrightarrow{\;ev_1\;} k \Big)$; in order to show that $\widehat{U}_q \supseteq \widetilde{\widetilde{U}}_q$ it is enough to check that $\widehat{U}_q \supseteq (q-1)^{-1} I'$, for the latter space generates $\widetilde{\widetilde{U}}_q$ (as a unital $k\big[q,q^{-1}\big]$–algebra). So let $x' \in I'$: then there exists $n \in \mathbb{N}$ such that $x := (q-1)^{-n} x' \in \widehat{U}_q \setminus (q-1)\,\widehat{U}_q$. Now $x' \in I'$ implies $\delta_1(x') \in (q-1)\,\widehat{U}_q$, hence $x' = \delta_1(x') + \epsilon(x') \in (q-1)\,\widehat{U}_q$: therefore $n > 0$; then $(q-1)^{-1} x' = (q-1)^{-1} \cdot (q-1)^n x \in (q-1)^{n-1}\,\widehat{U}_q$, so $(q-1)^{-1} x' \in \widehat{U}_q$, q.e.d.

<u>*Claim 2:*</u> *For* $x' \in \widetilde{U}_q$, *let* $x \in \widehat{U}_q \setminus (q-1)\,\widehat{U}_q$, $n \in \mathbb{N}$, *be such that* $x' = (q-1)^n x$. *Set* $\bar{x} := x \mod (q-1)\,\widehat{U}_q \in \widehat{U}_q \Big/ (q-1)\,\widehat{U}_q = U(\mathfrak{g})$. *Then* $\partial(\bar{x}) \leq n$.

Proof of Claim 2. By hypothesis $\delta_{n+1}(x') \in (q-1)^{n+1} {\widehat{U}_q}^{\otimes(n+1)}$, hence we have $\delta_{n+1}(x) \in (q-1)\,{\widehat{U}_q}^{\otimes(n+1)}$, so $\delta_{n+1}(\bar{x}) = 0$, i.e. $\bar{x} \in Ker\Big(\delta_{n+1} \colon U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\otimes(n+1)} \Big)$ (where $\mathfrak{g}$ is the Lie bialgebra such that $\widehat{U}_q \Big/ (q-1)\,\widehat{U}_q = U(\mathfrak{g})$). But this kernel equals the subspace $U(\mathfrak{g})_n := \big\{ \bar{y} \in U(\mathfrak{g}) \,\big|\, \partial(\bar{y}) \leq n \big\}$ (see e.g. [KT], §3.8), whence the claim follows, q.e.d.

Now consider the QFA $\big(F_q, \widehat{F}_q\big)$: thanks to *Claim 1.(a)* we have only to prove $\widehat{F}_q \supseteq \widetilde{\widetilde{F}}_q$.

Let $x' \in \widetilde{\widetilde{F}}_q$ be given; let $n \in \mathbb{N}$, $x \in \widetilde{F}_q \setminus (q-1)\widetilde{F}_q$ be such that $x' = (q-1)^n x$. Due to Propositions 2.5 and 2.7, $\big(F_q, \widetilde{F}_q\big)$ is a (G)QUEA (with semiclassical limit $U(\mathfrak{g}^*)$; then *Claim 2* gives $\partial(\bar{x}) \leq n$. Fix an ordered basis $\{b_\lambda\}_{\lambda\in\Lambda}$ of $\mathfrak{g}^*$, and a subset $\{x_\lambda\}_{\lambda\in\Lambda}$ of $\widetilde{F}_q$ such that $x_\lambda \mod (q-1)\,\widetilde{F}_q = b_\lambda$ for all λ ; in particular, since $\mathfrak{g}^* \subset Ker\big(\epsilon_{U(\mathfrak{g}^*)}\big)$ we can choose the x_λ's inside $\tilde{J} := (q-1)^{-1} J$, where $J := Ker\big(\epsilon \colon \widehat{F}_q \longrightarrow k\big[q,q^{-1}\big]\big)$: so $x_\lambda = (q-1)^{-1} x'_\lambda$ for some $x'_\lambda \in J$, for all $\lambda \in \Lambda$. Since $\partial(\bar{x}) \leq n$, i.e. $\bar{x} \in U(\mathfrak{g}^*)_n$, we can write $\bar{x}$ as a polynomial $P\big(\{b_\lambda\}_{\lambda\in\Lambda}\big)$ in the b_λ's of degree $d \leq n$; then $P\big(\{x_\lambda\}_{\lambda\in\Lambda}\big) \equiv x \mod (q-1)\,\widetilde{F}_q$, so we can assume $x = P\big(\{x_\lambda\}_{\lambda\in\Lambda}\big)$. Now we can write x as $x = \sum_{s=0}^d j_s (q-1)^{-s}$, where $j_s \in J^s$ is a homogeneous polynomial in the x'_λ's of degree s, and $j_d \neq 0$; then $x' = (q-1)^n x = \sum_{s=0}^d j_s (q-1)^{n-s} \in \widehat{F}_q$ (because $d \leq n$), q.e.d.

<u>*Claim 3:*</u> $\quad \widehat{U}_q{}^* = \widetilde{\big(\widehat{U}_q^*\big)^\vee}$.

Proof of Claim 3. This works like the identity $\widehat{F}_q = \widetilde{\widetilde{F}}_q$: the inclusion $\widehat{U}_q{}^* \subseteq \widetilde{\big(\widehat{U}_q^*\big)^\vee}$ is proved exactly like part *(a)* of *Claim 1*, and the reverse inclusion $\widehat{U}_q{}^* \supseteq \widetilde{\big(\widehat{U}_q^*\big)^\vee}$ is proved by the same argument used above to show that $\widehat{F}_q = \widetilde{\widetilde{F}}_q$, q.e.d.

Finally, the case of $\big(U_q, \widehat{U}_q\big)$. By Proposition 2.3, $\widetilde{U}_q$ is in perfect pairing with $\big(\widehat{U}_q^*\big)^\vee$; now, we can adapt the same proposition to the case of the (G)QUEA $\big(\widehat{U}_q^*\big)^\vee$ (instead of U_q) and the QFA $\widetilde{U}_q$ instead of $U_q{}^*$ (the same proof works, up to a few minor changes): then $\widetilde{\widetilde{U}}_q$ is in perfect pairing with $\widetilde{\big(\widehat{U}_q^*\big)^\vee}$. By *Claim 3*, the latter equals $\widehat{U}_q{}^*$, hence is also in perfect pairing with $\widehat{U}_q$. Now $\widetilde{\widetilde{U}}_q \subseteq \widehat{U}_q$, thanks to *Claim 1.(b)*: if the inclusion were strict, there would exist $\phi \in \widehat{U}_q{}^* \setminus \{0\}$ such that $\Big\langle \phi \, , \, \widetilde{\widetilde{U}}_q \Big\rangle = 0$: but this contradicts the non-degeneracy of the pairing between $\widetilde{\widetilde{U}}_q$ and $\widetilde{\big(\widehat{U}_q^*\big)^\vee} = \widehat{U}_q{}^*$, thus $\widetilde{\widetilde{U}}_q = \widehat{U}_q$, q.e.d. $\square$

§ 3 First example: SL_2, SL_n, and the semisimple case

3.1 The classical setting. Let $G := SL_2$. Its tangent Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ is generated by f, h, e (the *Chevalley generators*) with relations $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$. The formulae $\delta(f) = h \otimes f - f \otimes h$, $\delta(h) = 0$, $\delta(e) = h \otimes e - e \otimes h$, define a Lie cobracket on $\mathfrak{g}$ which gives it a structure of Lie bialgebra, which corresponds to a structure of Poisson group on G. These same formulae, when properly read, give a presentation of the co-Poisson Hopf algebra $U(\mathfrak{g})$ (with the standard Hopf structure).

On the other hand, $F\big[SL_2\big]$ is the unital associative commutative k–algebra with generators a, b, c, d and the relation $ad - bc = 1$, and Poisson Hopf structure given by

$$\Delta(a) = a \otimes a + b \otimes c\,,\ \ \Delta(b) = a \otimes b + b \otimes d\,,\ \ \Delta(c) = c \otimes a + d \otimes c\,,\ \ \Delta(d) = c \otimes b + d \otimes d$$
$$\epsilon(a) = 1\,,\ \ \epsilon(b) = 0\,,\ \ \epsilon(c) = 0\,,\ \ \epsilon(d) = 1\,, \qquad S(a) = d\,,\ \ S(b) = -b\,,\ \ S(c) = -c\,,\ \ S(d) = a$$
$$\{a,b\} = ba\,, \quad \{a,c\} = ca\,, \quad \{b,c\} = 0\,, \quad \{d,b\} = -bd\,, \quad \{d,c\} = -cd\,, \quad \{a,d\} = 2bc\,.$$

The dual Lie bialgebra $\mathfrak{g}^* = {\mathfrak{sl}_2}^*$ is the Lie algebra with generators $\mathrm{f}, \mathrm{h}, \mathrm{e}$, and relations $[\mathrm{h},\mathrm{e}] = \mathrm{e}$, $[\mathrm{h},\mathrm{f}] = \mathrm{f}$, $[\mathrm{e},\mathrm{f}] = 0$, with Lie cobracket given by $\delta(\mathrm{f}) = 2(\mathrm{f} \otimes \mathrm{h} - \mathrm{h} \otimes \mathrm{f})$, $\delta(\mathrm{h}) = \mathrm{e} \otimes \mathrm{f} - \mathrm{f} \otimes \mathrm{e}$, $\delta(\mathrm{e}) = 2(\mathrm{h} \otimes \mathrm{e} - \mathrm{e} \otimes \mathrm{h})$ (we choose as generators $\mathrm{f} := f^*$, $\mathrm{h} := h^*$, $\mathrm{e} := e^*$, where $\big\{f^*, h^*, e^*\big\}$ is the basis of ${\mathfrak{sl}_2}^*$ which is the dual of the basis $\{f,h,e\}$ of $\mathfrak{sl}_2$). This again yields also a presentation of $U\big({\mathfrak{sl}_2}^*\big)$. The simply connected algebraic Poisson group whose tangent Lie bialgebra is ${\mathfrak{sl}_2}^*$ can be realized as the group of pairs of matrices (the left subscript s meaning "simply connected")

$${}_sSL_2{}^* = \left\{ \left(\begin{pmatrix} t^{-1} & 0 \\ y & t \end{pmatrix}, \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} \right) \,\middle|\, x, y \in k, t \in k \setminus \{0\} \right\} \leq SL_2 \times SL_2\ ;$$

this group has centre $Z := \big\{(I,I), (-I,-I)\big\}$, so there is only one other (Poisson) group sharing the same Lie (bi)algebra, namely the quotient ${}_aSL_2{}^* := {}_sSL_2{}^* \big/ Z$ (the adjoint of ${}_sSL_2{}^*$, as the left subscript a means). Therefore $F\big[{}_sSL_2{}^*\big]$ is the unital associative commutative k–algebra with generators x, $z^{\pm 1}$, y, with Poisson Hopf structure given by

$$\Delta(x) = x \otimes z^{-1} + z \otimes x\,, \qquad \Delta\big(z^{\pm 1}\big) = z^{\pm 1} \otimes z^{\pm 1}\,, \qquad \Delta(y) = y \otimes z^{-1} + z \otimes y$$
$$\epsilon(x) = 0\,, \quad \epsilon\big(z^{\pm 1}\big) = 1\,, \quad \epsilon(y) = 0\,, \qquad S(x) = -x\,, \quad S\big(z^{\pm 1}\big) = z^{\mp 1}\,, \quad S(y) = -y$$
$$\{x,y\} = \big(z^2 - z^{-2}\big)/2\,, \qquad \big\{z^{\pm 1}, x\big\} = \pm x z^{\pm 1}\,, \qquad \big\{z^{\pm 1}, y\big\} = \mp z^{\pm 1} y$$

(N.B.: with respect to this presentation, we have $\mathrm{f} = \partial_y\big|_e$, $\mathrm{h} = \frac{z}{2}\partial_z\big|_e$, $\mathrm{e} = \partial_x\big|_e$, where e is the identity element of ${}_sSL_2{}^*$). Moreover, $F\big[{}_aSL_2{}^*\big]$ can be identified with the Poisson Hopf subalgebra of $F\big[{}_sSL_2{}^*\big]$ spanned by products of an even number of generators — i.e. monomials of even degree: this is generated, as a unital subalgebra, by xz, $z^{\pm 2}$, and $z^{-1}y$.

In general, we shall consider $\mathfrak{g} = \mathfrak{g}^\tau$ a semisimple Lie algebra, endowed with the Lie cobracket — depending on the parameter τ — given in [Ga1], §1.3; in the following we shall also retain from [*loc. cit.*] all the notation we need: in particular, we denote by Q, resp. P, the root lattice, resp. the weight lattice, of $\mathfrak{g}$, and by r the rank of $\mathfrak{g}$.

3.2 The (G)QUEA's $\big(U_q(\mathfrak{g}), \widehat{U}_q(\mathfrak{g})\big)$**.** We turn now to quantum groups, starting with the $\mathfrak{sl}_2$ case. Let $U_q(\mathfrak{g}) = U_q\big(\mathfrak{sl}_2\big)$ be the associative unital $k(q)$–algebra with (*Chevalley-like*) generators F, $K^{\pm 1}$, E, and relations

$$K K^{-1} = 1 = K^{-1} K\,,\ K^{\pm 1} F = q^{\mp 2} F K^{\pm 1}\,,\ K^{\pm 1} E = q^{\pm 2} E K^{\pm 1}\,,\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\,.$$

This is a Hopf algebra, with Hopf structure given by

$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F\,, \qquad \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, \qquad \Delta(E) = E \otimes 1 + K \otimes E$$
$$\epsilon(F) = 0\,,\ \epsilon\big(K^{\pm 1}\big) = 1\,,\ \epsilon(E) = 0\,, \qquad S(F) = -FK,\ S\big(K^{\pm 1}\big) = K^{\mp 1},\ S(E) = -K^{-1}E\,.$$

Then let $\widehat{U}_q(\mathfrak{g})$ be the $k\big[q,q^{-1}\big]$–subalgebra of $U_q(\mathfrak{g})$ generated by F, $H := \dfrac{K-1}{q-1}$, $\varGamma := \dfrac{K-K^{-1}}{q-q^{-1}}$, $K^{\pm 1}$, E. From the definition of $U_q(\mathfrak{g})$ one gets a presentation of $\widehat{U}_q(\mathfrak{g})$ as the associative unital algebra with generators F, H, $\varGamma$, $K^{\pm 1}$, E and relations

$$KK^{-1} = 1 = K^{-1}K\,, \quad K^{\pm 1}H = HK^{\pm 1}\,, \quad K^{\pm 1}\varGamma = \varGamma K^{\pm 1}\,, \quad H\varGamma = \varGamma H$$
$$(q-1)H = K-1\,, \quad \big(q-q^{-1}\big)\varGamma = K-K^{-1}\,, \quad H\big(1+K^{-1}\big) = \big(1+q^{-1}\big)\varGamma\,, \quad EF-FE = \varGamma$$
$$K^{\pm 1}F = q^{\mp 2}FK^{\pm 1}\,, \quad HF = q^{-2}FH - (q+1)F\,, \quad \varGamma F = q^{-2}F\varGamma - \big(q+q^{-1}\big)F$$
$$K^{\pm 1}E = q^{\pm 2}EK^{\pm 1}\,, \quad HE = q^{+2}EH + (q+1)E\,, \quad \varGamma E = q^{+2}E\varGamma + \big(q+q^{-1}\big)E$$

and with a Hopf structure given by the same formulae as above for F, $K^{\pm 1}$, and E plus

$$\Delta(\varGamma) = \varGamma \otimes K + K^{-1} \otimes \varGamma\,, \qquad \epsilon(\varGamma) = 0\,, \qquad S(\varGamma) = -\varGamma$$
$$\Delta(H) = H \otimes 1 + K \otimes H\,, \qquad \epsilon(H) = 0\,, \qquad S(H) = -K^{-1}H\,.$$

Note also that $K = 1 + (q-1)H$ and $K^{-1} = K - \big(q-q^{-1}\big)\varGamma = 1 + (q-1)H - \big(q-q^{-1}\big)\varGamma$, hence $\widehat{U}_q(\mathfrak{g})$ is generated even by F, H, $\varGamma$ and E alone. It is easy to check that $\big(U_q(\mathfrak{g}), \widehat{U}_q(\mathfrak{g})\big)$ is a (G)QUEA, whose semiclassical limit is $U(\mathfrak{g}) = U\big(\mathfrak{sl}_2\big)$: the generators F, $K^{\pm 1}$, H, $\varGamma$, E respectively map to f, 1, h, h, $e \in U(\mathfrak{sl}_2)$. Finally, we record that

$$\widehat{U}_q(\mathfrak{g}) \ = \ k\big[q,q^{-1}\big]\text{–span of } \Big\{ F^a H^b \varGamma^c E^d \,\Big|\, a,b,c,d \in \mathbb{N} \Big\}\,; \tag{3.1}$$

In the general case of semisimple $\mathfrak{g}$, let $U_q(\mathfrak{g})$ be the Lusztig-like quantum group — over $k\big[q,q^{-1}\big]$ — associated to $\mathfrak{g} = \mathfrak{g}^\tau$ as in [Ga1], namely $U_q(\mathfrak{g}) := U_{q,\varphi}^M(\mathfrak{g})$ with respect to the notation in [*loc. cit.*], where M is any intermediate lattice such that $Q \leq M \leq P$ (this is just a matter of choice, of the type mentioned in the statement of Theorem 2.1*(c)*): this is a Hopf algebra over $k(q)$, generated by some elements F_i, M_i, E_i for $i = 1, \dots, r =: \mathit{rank}(\mathfrak{g})$. Then let $\widehat{U}_q(\mathfrak{g})$ be the unital $k\big[q,q^{-1}\big]$–subalgebra of $U_q(\mathfrak{g})$ generated by the elements F_i, $H_i := \dfrac{M_i - 1}{q-1}$, $\varGamma_i := \dfrac{K_i - K_i^{-1}}{q-q^{-1}}$, $M_i^{\pm 1}$, E_i, where the $K_i = M_{\alpha_i}$ are suitable product of M_j's, defined as in [Ga1], §2.2 (whence $K_i, K_i^{-1} \in \widehat{U}_q(\mathfrak{g})$): then again $\big(U_q(\mathfrak{g}), \widehat{U}_q(\mathfrak{g})\big)$ is a (G)QUEA, with semiclassical limit $U(\mathfrak{g})$. For later use, we record the following: from [Ga1], §§2.5, 3.3, we get that $\widehat{U}_q(\mathfrak{g})$ is the $k\big[q,q^{-1}\big]$–span of the set of monomials

$$\bigg\{ \prod_{\alpha \in R^+} F_\alpha^{f_\alpha} \cdot \prod_{i=1}^n H_i^{t_i} \cdot \prod_{i=1}^n \varGamma_i^{c_i} \cdot \prod_{\alpha \in R^+} E_\alpha^{e_\alpha} \,\bigg|\, f_\alpha, t_i, c_j, e_\alpha \in \mathbb{N} \ \forall\, \alpha \in R^+,\ i = 1, \dots, n \bigg\}\,; \tag{3.2}$$

hereafter, R^+ is the set of positive roots of $\mathfrak{g}$, each E_α, resp. F_α, is a root vector attached to the positive root $\alpha \in R^+$, resp. the negative root $\alpha \in (-R^+)$, and the products of factors indexed by R^+ are ordered with respect to a fixed convex order of R^+ (see [Ga1]).

3.3 Computation of $\widetilde{U}_q(\mathfrak{g})$ and specialization $\widetilde{U}_q(\mathfrak{g}) \xrightarrow{q \to 1} F[G^*]$. We begin with the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. From the definition of $\widehat{U}_q(\mathfrak{g}) = \widehat{U}_q(\mathfrak{sl}_2)$ we have, for any $n \in \mathbb{N}$,

$$\delta_n(E) = (\mathrm{id} - \epsilon)^{\otimes n}\big(\Delta^n(E)\big) = (\mathrm{id} - \epsilon)^{\otimes n}\left(\sum_{s=1}^n K^{\otimes(s-1)} \otimes E \otimes 1^{\otimes(n-s)}\right) =$$
$$= (\mathrm{id} - \epsilon)^{\otimes n}\big(K^{\otimes(n-1)} \otimes E\big) = (K-1)^{\otimes(n-1)} \otimes E = (q-1)^{n-1} \cdot H^{\otimes(n-1)} \otimes E$$

from which $\delta_n\big((q-1)E\big) \in (q-1)^n \widehat{U}_q(\mathfrak{g}) \setminus (q-1)^{n+1}\widehat{U}_q(\mathfrak{g})$, whence $(q-1)E \in \widetilde{U}_q(\mathfrak{g})$, whereas $E \notin \widetilde{U}_q(\mathfrak{g})$. Similarly, $(q-1)F \in \widetilde{U}_q(\mathfrak{g})$, whilst $F \notin \widetilde{U}_q(\mathfrak{g})$. As for generators H, $\varGamma$, $K^{\pm 1}$, we have $\Delta^n(H) = \sum_{s=1}^n K^{\otimes(s-1)} \otimes H \otimes 1^{\otimes(n-s)}$, $\Delta^n\big(K^{\pm 1}\big) = \big(K^{\pm 1}\big)^{\otimes n}$, $\Delta^n(\varGamma) = \sum_{s=1}^n K^{\otimes(s-1)} \otimes \varGamma \otimes \big(K^{-1}\big)^{\otimes(n-s)}$, hence for $\delta_n = (\mathrm{id} - \epsilon)^{\otimes n} \circ \Delta^n$ we have

$$\delta_n(H) = (q-1)^{n-1} \cdot H^{\otimes n}\,, \qquad \delta^n\big(K^{-1}\big) = (q-1)^n \cdot \big(-K^{-1}H\big)^{\otimes n}$$
$$\delta^n(K) = (q-1)^n \cdot H^{\otimes n}\,, \quad \delta^n(\varGamma) = (q-1)^{n-1} \cdot \sum_{s=1}^n (-1)^{n-s} H^{\otimes(s-1)} \otimes \varGamma \otimes \big(HK^{-1}\big)^{\otimes(n-s)}$$

for all $n \in \mathbb{N}$, so that $(q-1)H$, $(q-1)\varGamma$, $K^{\pm 1} \in \widetilde{U}_q(\mathfrak{g}) \setminus (q-1)\widetilde{U}_q(\mathfrak{g})$. Therefore $\widetilde{U}_q(\mathfrak{g})$ contains the subalgebra U' generated by $(q-1)F$, K, K^{-1}, $(q-1)H$, $(q-1)\varGamma$, $(q-1)E$. On the other hand, using (3.1) a thorough — but straightforward — computation along the same lines as above shows that any element in $\widetilde{U}_q(\mathfrak{g})$ does necessarily lie in U' (details are left to the reader: everything follows from definitions and the formulae above for Δ^n). Thus $\widetilde{U}_q(\mathfrak{g})$ is nothing but the subalgebra of $\widehat{U}_q(\mathfrak{g})$ generated by $\dot{F} := (q-1)F$, K, K^{-1}, $\dot{H} := (q-1)H$, $\dot{\varGamma} := (q-1)\varGamma$, $\dot{E} := (q-1)E$; notice also that the generator $\dot{H}$ is unnecessary, for $\dot{H} = K - 1$. As a consequence, $\widetilde{U}_q(\mathfrak{g})$ can be presented as the unital associative $k\big[q, q^{-1}\big]$–algebra with generators $\dot{F}$, $\dot{\varGamma}$, $K^{\pm 1}$, $\dot{E}$ and relations

$$KK^{-1} = 1 = K^{-1}K, \;\; K^{\pm 1}\dot{\varGamma} = \dot{\varGamma}K^{\pm 1}, \;\; \big(1+q^{-1}\big)\dot{\varGamma} = K - K^{-1}, \;\; \dot{E}\dot{F} - \dot{F}\dot{E} = (q-1)\dot{\varGamma}$$
$$(K-1)\big(1+K^{-1}\big) = \big(1+q^{-1}\big)\dot{\varGamma}\,, \quad K^{\pm 1}\dot{F} = q^{\mp 2}\dot{F}K^{\pm 1}\,, \quad K^{\pm 1}\dot{E} = q^{\pm 2}\dot{E}K^{\pm 1}$$
$$\dot{\varGamma}\dot{F} = q^{-2}\dot{F}\dot{\varGamma} - (q-1)\big(q+q^{-1}\big)\dot{F}\,, \qquad \dot{\varGamma}\dot{E} = q^{+2}\dot{E}\dot{\varGamma} + (q-1)\big(q+q^{-1}\big)\dot{E}$$

with Hopf structure given by

$$\begin{array}{lll}
\Delta\big(\dot{F}\big) = \dot{F} \otimes K^{-1} + 1 \otimes \dot{F}\,, & \epsilon\big(\dot{F}\big) = 0\,, & S\big(\dot{F}\big) = -\dot{F}K \\
\Delta\big(\dot{\varGamma}\big) = \dot{\varGamma} \otimes K + K^{-1} \otimes \dot{\varGamma}\,, & \epsilon\big(\dot{\varGamma}\big) = 0\,, & S\big(\dot{\varGamma}\big) = -\dot{\varGamma} \\
\Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, & \epsilon\big(K^{\pm 1}\big) = 1\,, & S\big(K^{\pm 1}\big) = K^{\mp 1} \\
\Delta\big(\dot{E}\big) = \dot{E} \otimes 1 + K \otimes \dot{E}\,, & \epsilon\big(\dot{E}\big) = 0\,, & S\big(\dot{E}\big) = -K^{-1}\dot{E}\,.
\end{array}$$

When $q \to 1$, an easy direct computation shows that this gives a presentation of the function algebra $F\big[{}_aSL_2{}^*\big]$, and the Poisson structure that $F\big[{}_aSL_2{}^*\big]$ inherits from this

quantization process is exactly the one coming from the Poisson structure on ${}_aSL_2{}^*$: in fact, there is a Poisson Hopf algebra isomorphism

$$\widetilde{U}_q(\mathfrak{g}) \Big/ (q-1)\, \widetilde{U}_q(\mathfrak{g}) \xrightarrow{\ \cong\ } F\big[{}_aSL_2{}^*\big] \quad \Big(\subseteq F\big[{}_sSL_2{}^*\big] \Big)$$

given by: $\dot{E} \mod (q-1) \mapsto xz$, $K^{\pm 1} \mod (q-1) \mapsto z^{\pm 2}$, $\dot{H} \mod (q-1) \mapsto z^2 - 1$, $\dot{\varGamma} \mod (q-1) \mapsto \big(z^2 - z^{-2}\big) \Big/ 2$, $\dot{F} \mod (q-1) \mapsto z^{-1} y$. In other words, $\widetilde{U}_q(\mathfrak{g})$ specializes to $F\big[{}_aSL_2{}^*\big]$ *as a Poisson Hopf algebra*, as predicted by Theorem 2.1.

Note that in this case we got the *adjoint* Poisson group dual of $G = SL_2$, that is ${}_aSL_2{}^*$; a different choice of the initial (G)QUEA leads us to get the *simply connected* one, i.e. ${}_sSL_2{}^*$, as follows. Start from a "simply connected" version of $U_q(\mathfrak{g})$, obtained from the previous one by adding a square root of K, call it L, and its inverse, and do the same when defining $\widehat{U}_q(\mathfrak{g})$. Then the new pair $\big(U_q(\mathfrak{g}), \widehat{U}_q(\mathfrak{g})\big)$ is again a quantization of $U(\mathfrak{g})$, and $\widetilde{U}_q(\mathfrak{g})$ is like above but for the presence of the new generator L, and the same is when specializing q at 1: thus we get the function algebra of a Poisson group which is a double covering of ${}_aSL_2{}^*$, that is exactly ${}_sSL_2{}^*$. So changing the choice of the (G)QUEA quantizing $\mathfrak{g}$ we get two different QFA's, one for each of the two connected Poisson algebraic groups dual of SL_2, i.e. having tangent Lie bialgebra $\mathfrak{sl}_2{}^*$; this shows the dependence of the dual group G^*, mentioned in Theorem 2.1, on the choice of the (G)QUEA (for fixed $\mathfrak{g}$).

With a bit more careful study, exploiting the analysis in [Ga1], one can treat the general case too: we sketch briefly our arguments — restricting to the simply laced case, to simplify the exposition — leaving to the reader the (straightforward) task of filling details.

So now let $\mathfrak{g} = \mathfrak{g}^\tau$ be a semisimple Lie algebra, as in §3.1, and let $\big(U_q(\mathfrak{g}), \widehat{U}_q(\mathfrak{g})\big)$ be the (G)QUEA introduced in §3.2: our aim again is to compute the QFA $\big(U_q(\mathfrak{g}), \widetilde{U}_q(\mathfrak{g})\big)$.

The same computations as for $\mathfrak{g} = \mathfrak{sl}(2)$ show that $\delta_n(H_i) = (q-1)^{n-1} \cdot H_i^{\otimes n}$ and $\delta^n(\varGamma_i) = (q-1)^{n-1} \cdot \sum_{s=1}^n (-1)^{n-s} H_i^{\otimes (s-1)} \otimes \varGamma_i \otimes \big(H_i K_i^{-1}\big)^{\otimes (n-s)}$, which gives

$$\dot{H}_i := (q-1) H_i \in \widetilde{U}_q(\mathfrak{g}) \setminus (q-1)\, \widetilde{U}_q(\mathfrak{g}) \quad \text{and} \quad \dot{\varGamma}_i := (q-1)\, \varGamma_i \in \widetilde{U}_q(\mathfrak{g}) \setminus (q-1)\, \widetilde{U}_q(\mathfrak{g}) \,.$$

As for root vectors, let $\dot{E}_\gamma := (q-1) E_\gamma$ and $\dot{F}_\gamma := (q-1) F_\gamma$ for all $\gamma \in R^+$: then acting as in [Ga1], §5.16, we can prove that $E_\alpha \notin \widetilde{U}_q(\mathfrak{g})$ but $\dot{E}_\alpha \in \widetilde{U}_q(\mathfrak{g}) \setminus (q-1)\, \widetilde{U}_q(\mathfrak{g})$. In fact, let $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ be quantum Borel subalgebras, and $\mathfrak{U}_{\varphi,\geq}^M$, $\mathcal{U}_{\varphi,\geq}^M$, $\mathfrak{U}_{\varphi,\leq}^M$, $\mathcal{U}_{\varphi,\leq}^M$ their $k\big[q,q^{-1}\big]$–subalgebras defined in [Ga1], §2: then both $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ are Hopf subalgebras of $U_q(\mathfrak{g})$; in addition, letting M' be the lattice between Q and P dual of M (in the sense of [Ga1], §1.1, there exists a $k(q)$–valued perfect Hopf pairing between $U_q(\mathfrak{b}_\pm)$ and $U_q(\mathfrak{b}_\mp)$ — one built up on M and the other on M' — such that $\mathfrak{U}_{\varphi,\geq}^M = \Big(\mathcal{U}_{\varphi,\leq}^{M'}\Big)^\dagger$, $\mathfrak{U}_{\varphi,\leq}^M = \Big(\mathcal{U}_{\varphi,\geq}^{M'}\Big)^\dagger$, $\mathcal{U}_{\varphi,\geq}^M = \Big(\mathfrak{U}_{\varphi,\leq}^{M'}\Big)^\dagger$, and $\mathcal{U}_{\varphi,\leq}^M = \Big(\mathfrak{U}_{\varphi,\geq}^{M'}\Big)^\dagger$. Now, $(q-q^{-1}) E_\alpha \in \mathcal{U}_{\varphi,\geq}^M = \Big(\mathfrak{U}_{\varphi,\leq}^{M'}\Big)^\dagger$, hence — since $\mathfrak{U}_{\varphi,\leq}^{M'}$ is an algebra — we have $\Delta\Big((q-q^{-1}) E_\alpha\Big) \in \Big(\mathfrak{U}_{\varphi,\leq}^{M'} \otimes \mathfrak{U}_{\varphi,\leq}^{M'}\Big)^\dagger =$

$\left(\mathfrak{U}^{M'}_{\varphi,\leq}\right)^{\dagger} \otimes \left(\mathfrak{U}^{M'}_{\varphi,\leq}\right)^{\dagger} = \mathcal{U}^{M}_{\varphi,\geq} \otimes \mathcal{U}^{M}_{\varphi,\geq}$. Therefore, by definition of $\mathcal{U}^{M}_{\varphi,\geq}$ and by the PBW theorem for it and for $\mathfrak{U}^{M'}_{\varphi,\leq}$ (cf. [Ga1], §2.5) we have that $\Delta\Big(\big(q-q^{-1}\big)E_\alpha\Big)$ is a $k\big[q,q^{-1}\big]$–linear combination like $\Delta\Big(\big(q-q^{-1}\big)E_\alpha\Big) = \sum_r A_r^{(1)} \otimes A_r^{(2)}$ in which the $A_r^{(j)}$'s are monomials in the M_j's and in the $\overline{E}_\gamma$'s, where $\overline{E}_\gamma := \big(q-q^{-1}\big)E_\gamma$ for all $\gamma \in R^+$: iterating, we find that $\Delta^\ell\Big(\big(q-q^{-1}\big)E_\alpha\Big)$ is a $k\big[q,q^{-1}\big]$–linear combination

$$\Delta^\ell\Big(\big(q-q^{-1}\big)E_\alpha\Big) = \sum_r A_r^{(1)} \otimes A_r^{(2)} \otimes \cdots \otimes A_r^{(\ell)} \tag{3.3}$$

in which the $A_r^{(j)}$'s are again monomials in the M_j's and in the $\overline{E}_\gamma$'s. Now, we distinguish two cases: either $A_r^{(j)}$ does contain some $\overline{E}_\gamma$ $\big(\in \big(q-q^{-1}\big)\,\widehat{U}_q(\mathfrak{g})$, thus $\epsilon\Big(A_r^{(j)}\Big) = A_r^{(j)} \in (q-1)\widehat{U}_q(\mathfrak{g})$ whence $(id-\epsilon)\Big(A_r^{(j)}\Big) = 0$; or $A_r^{(j)}$ does not contain any $\overline{E}_\gamma$ and is only a monomial in the M_t's, say $A_r^{(j)} = \prod_{t=1}^n M_t^{m_t}$: then $(id-\epsilon)\Big(A_r^{(j)}\Big) = \prod_{t=1}^n M_t^{m_t} - 1 = \prod_{t=1}^n \big((q-1)\,H_t + 1\big)^{m_t} - 1 \in (q-1)\,\widehat{U}_q(\mathfrak{g})$. In addition, for some "Q–grading reasons" (as in [Ga1], §5.16), in each one of the summands in (3.3) the sum of all the γ's such that the (rescaled) root vectors $\overline{E}_\gamma$ occur in any of the factors $A_r^{(1)}$, $A_r^{(2)}$, ..., $A_r^{(n)}$ must be equal to α: therefore, in each of these summands at least one factor $\overline{E}_\gamma$ does occur. The upset is that $\delta_\ell\big(\overline{E}_\alpha\big) \in \big(1+q^{-1}\big)(q-1)^\ell\,\widehat{U}_q(\mathfrak{g})^{\otimes \ell}$ (the factor $\big(1+q^{-1}\big)$ being there because at least one rescaled root vector $\overline{E}_\gamma$ occurs in each summand of $\delta_\ell\big(\overline{E}_\alpha\big)$, thus providing a coefficient $\big(q-q^{-1}\big)$ the term $\big(1+q^{-1}\big)$ is factored out of), whence $\delta_\ell\big(\dot{E}_\alpha\big) \in (q-1)^\ell\,\widehat{U}_q(\mathfrak{g})^{\otimes \ell}$. More precisely, we have also $\delta_\ell\big(\dot{E}_\alpha\big) \notin (q-1)^{\ell+1}\,\widehat{U}_q(\mathfrak{g})^{\otimes \ell}$, for we can easily check that $\Delta^\ell\big(\dot{E}_\alpha\big)$ is the sum of $M_\alpha \otimes M_\alpha \otimes \cdots \otimes M_\alpha \otimes \dot{E}_\alpha$ plus other summands which are $k\big[q,q^{-1}\big]$–linearly independent of this first term: but then $\delta_\ell\big(\dot{E}_\alpha\big)$ is the sum of $(q-1)^{\ell-1} H_\alpha \otimes H_\alpha \otimes \cdots \otimes H_\alpha \otimes \dot{E}_\alpha$ (where $H_\alpha := \frac{M_\alpha - 1}{q-1}$ is equal to a $k\big[q,q^{-1}\big]$–linear combination of products of M_j's and H_t's) plus other summands which are $k\big[q,q^{-1}\big]$–linearly independent of the first one, and since $H_\alpha \otimes H_\alpha \otimes \cdots \otimes H_\alpha \otimes \dot{E}_\alpha \notin (q-1)^2\,\widehat{U}_q(\mathfrak{g})^{\otimes \ell}$ we can conclude as claimed. Therefore $\delta_\ell\big(\dot{E}_\alpha\big) \in (q-1)^\ell\,\widehat{U}_q(\mathfrak{g})^{\otimes \ell} \setminus (q-1)^{\ell+1}\,\widehat{U}_q(\mathfrak{g})^{\otimes \ell}$, whence we get

$$\dot{E}_\alpha := (q-1)E_\alpha \in \widetilde{U}_q(\mathfrak{g}) \setminus (q-1)\,\widetilde{U}_q(\mathfrak{g}) \qquad \forall\ \alpha \in R^+ .$$

An entirely similar analysis yields also

$$\dot{F}_\alpha := (q-1)F_\alpha \in \widetilde{U}_q(\mathfrak{g}) \setminus (q-1)\,\widetilde{U}_q(\mathfrak{g}) \qquad \forall\ \alpha \in R^+ .$$

Summing up, we have found that $\widetilde{U}_q(\mathfrak{g})$ contains for sure the subalgebra U' generated by $\dot{F}_\alpha$, $\dot{H}_i$, $\dot{\varGamma}_i$, $\dot{E}_\alpha$ for all $\alpha \in R^+$ and all $i = 1,\dots,n$. On the other hand, using

(3.2) a thorough — but straightforward — computation along the same lines as above shows that any element in $\widetilde{U}_q(\mathfrak{g})$ must lie in U' (details are left to the reader). Thus finally $\widetilde{U}_q(\mathfrak{g}) = U'$, so we have a concrete description of $\widetilde{U}_q(\mathfrak{g})$.

Now compare $U' = \widetilde{U}_q(\mathfrak{g})$ with the algebra $\mathcal{U}_\varphi^{M}(\mathfrak{g})$ in [Ga1], §3.4 (for $\varphi = 0$), the latter being just the $k\big[q,q^{-1}\big]$–subalgebra of $U_q(\mathfrak{g})$ generated by the set $\big\{ \overline{F}_\alpha, M_i, \overline{E}_\alpha \,\big|\, \alpha \in R^+, i = 1, \dots, n \big\}$. First of all, by definition, we have $\mathcal{U}_\varphi^{M}(\mathfrak{g}) \subseteq U' = \widetilde{U}_q(\mathfrak{g})$; moreover,

$$\dot{F}_\alpha \equiv \frac{1}{2}\overline{F}_\alpha \,, \quad \dot{E}_\alpha \equiv \frac{1}{2}\overline{E}_\alpha \,, \quad \dot{\varGamma}_i \equiv \frac{1}{2}\big(K_i - K_i^{-1}\big) \mod (q-1)\,\mathcal{U}_\varphi^{M}(\mathfrak{g}) \qquad \forall\ \alpha,\ \forall\ i\,.$$

Then we have

$$\widetilde{U}_1(\mathfrak{g}) := \widetilde{U}_q(\mathfrak{g}) \Big/ (q-1)\,\widetilde{U}_q(\mathfrak{g}) \;=\; \mathcal{U}_\varphi^{M}(\mathfrak{g}) \Big/ (q-1)\,\mathcal{U}_\varphi^{M}(\mathfrak{g}) \;\cong\; F\big[G_M^*\big]$$

where G_M^* is the Poisson group dual of $G = G^\tau$ with centre $Z(G_M^*) \cong M/Q$ and fundamental group $\pi_1(G_M^*) \cong P/M$, and the isomorphism (of Poisson Hopf algebras) on the right is given by [Ga1], Theorem 7.4 (see also references therein for the original statement and proof of this result). In other words, $\widetilde{U}_q(\mathfrak{g})$ specializes to $F\big[G_M^*\big]$ *as a Poisson Hopf algebra*, as prescribed by Theorem 2.1. By the way, notice also that in the present case the dependence of $\widetilde{U}_1(\mathfrak{g})$ from the choice of the initial (G)QUEA (for fixed $\mathfrak{g}$) — mentioned in the last part of the statement of part *(c)* of Theorem 2.1 — is evident.

By the way, the previous discussion applies as well to the case of $\mathfrak{g}$ *an untwisted affine Kac-Moody algebra*: one just has to substitute any quotation from [Ga1] — referring to a some result about *finite* Kac-Moody algebras — with a similar quotation from [Ga4] — referring to the corresponding analogous result about untwisted *affine* Kac-Moody algebras.

3.4 The identity $\widetilde{\widetilde{U}}_q(\mathfrak{g}) = \widehat{U}_q(\mathfrak{g})$. In the present section we check that the first half of part *(b)* of Theorem 2.1 (*"The functors $\big(U_q, \widehat{U}_q\big) \longmapsto \big(U_q, \widetilde{U}_q\big)$ and $\big(F_q, \widehat{F}_q\big) \longmapsto \big(F_q, \widetilde{F}_q\big)$ are inverse of each other"*) is verified. Of course, we start once again from $\mathfrak{g} = \mathfrak{sl}_2$.

Since $\epsilon\big(\dot{F}\big) = \epsilon\big(\dot{H}\big) = \epsilon\big(\dot{\varGamma}\big) = \epsilon\big(\dot{E}\big) = 0$, the ideal $J := Ker\big(\epsilon : \widetilde{U}_q(\mathfrak{g}) \to k\big[q,q^{-1}\big]\,\big)$ is generated by $\dot{F}$, $\dot{H}$, $\dot{\varGamma}$, and $\dot{E}$. This implies that J is the $k\big[q,q^{-1}\big]$–span of $\Big\{ \dot{F}^\varphi \dot{H}^\kappa \dot{\varGamma}^\gamma \dot{E}^\eta \,\Big|\, (\varphi,\kappa,\gamma,\eta) \in \mathbb{N}^4 \setminus \{(0,0,0,0)\} \Big\}$. Now $I := Ker\Big(\widetilde{U}_q(\mathfrak{g}) \xrightarrow{\ \epsilon\ } k\big[q,q^{-1}\big] \xrightarrow{\ ev_1\ } k\Big) = J + (q-1)\cdot\widetilde{U}_q(\mathfrak{g})$, therefore we get that $\widetilde{\widetilde{U}}_q(\mathfrak{g}) := \sum_{n\geq 0} \Big((q-1)^{-1} I\Big)^n$ is generated, as a unital $k\big[q,q^{-1}\big]$-subalgebra of $U_q(\mathfrak{g})$, by the elements $(q-1)^{-1}\dot{F} = F$, $(q-1)^{-1}\dot{H} = H$, $(q-1)^{-1}\dot{\varGamma} = \varGamma$, $(q-1)^{-1}\dot{E} = E$, hence it coincides with $\widehat{U}_q(\mathfrak{g})$, q.e.d.

An entirely similar analysis works in the "adjoint" case as well; and also, *mutatis mutandis*, for the general semisimple case.

3.5 The QFA $\big(F_q[G], \widehat{F}_q[G]\big)$. In this and the following sections we pass to look at Theorem 2.1 the other way round: namely, we start from QFA's and produce (G)QUEA's.

So to begin we introduce the relevant QFA's; starting from $G = SL_n$ for which an especially explicit description of the QFA is available.

Consider $G := SL_n$ with the standard Poisson structure. Let $\widehat{F}_q\big[SL_n\big]$ be the unital associative $k\big[q,q^{-1}\big]$–algebra generated by $\{\, \rho_{ij} \mid i,j = 1, \dots, n \,\}$ with relations

$$\begin{aligned} \rho_{ij}\rho_{ik} = q\, \rho_{ik}\rho_{ij} \;, \qquad \rho_{ik}\rho_{hk} = q\, \rho_{hk}\rho_{ik} & \qquad \forall\, j<k,\; i<h \\ \rho_{il}\rho_{jk} = \rho_{jk}\rho_{il} \;, \qquad \rho_{ik}\rho_{jl} - \rho_{jl}\rho_{ik} = \big(q-q^{-1}\big)\; \rho_{il}\rho_{jk} & \qquad \forall\, i<j,\; k<l \\ det_q(\rho_{ij}) := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} \rho_{1,\sigma(1)}\rho_{2,\sigma(2)} \cdots \rho_{n,\sigma(n)} = 1 \,. & \end{aligned}$$

This is a Hopf algebra, with comultiplication, counit and antipode given by

$$\Delta(\rho_{ij}) = \sum_{k=1}^n \rho_{ik} \otimes \rho_{kj} \;,\; \epsilon(\rho_{ij}) = \delta_{ij} \;,\; S(\rho_{ij}) = (-q)^{j-i} det_q\left((\rho_{hk})_{h \neq j}^{k \neq i} \right) \;\forall\, i,j = 1, \dots, n \,.$$

Let also $F_q\big[SL_n\big] := k(q) \otimes_{k[q,q^{-1}]} \widehat{F}_q\big[SL_n\big]$. Then $\Big(F_q\big[SL_n\big], \widehat{F}_q\big[SL_n\big] \Big)$ is a QFA, with $\widehat{F}_q\big[SL_n\big] \xrightarrow{\;q \rightarrow 1\;} F\big[SL_n\big]$.

3.6 Computation of $\widetilde{F}_q[G]$ and specialization $\widetilde{F}_q[G] \xrightarrow{\;q \rightarrow 1\;} U(\mathfrak{g}^*)$. In this section we go and compute $\widetilde{F}_q[G]$ and its semiclassical limit (i.e. its specialization at $q = 1$). To begin with, the set of ordered monomials

$$M := \left\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} \rho_{hk}^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \,\middle|\, N_{st} \in \mathbb{N} \;\forall\, s,t \,;\, min\big\{ N_{1,1}, \dots, N_{n,n} \big\} = 0 \right\}$$

is a $k\big[q,q^{-1}\big]$–basis of $\widehat{F}_q\big[SL_n\big]$ and a $k(q)$–basis of $F_q\big[SL_n\big]$ (cf. [Ga2], Theorem 7.4: it deals only with $F_q\big[SL_n\big]$, but clearly holds for $\widehat{F}_q\big[SL_n\big]$ as well): then also the set

$$M' := \left\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} \big(\rho_{hk} - 1\big)^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \,\middle|\, N_{st} \in \mathbb{N} \;\forall\, s,t \,;\, min\big\{ N_{1,1}, \dots, N_{n,n} \big\} = 0 \right\}$$

is a $k\big[q,q^{-1}\big]$–basis of $\widehat{F}_q\big[SL_n\big]$ and a $k(q)$–basis of $F_q\big[SL_n\big]$. Then, from the definition of the counit, it follows that

$$M' \setminus \{1\} \quad \text{is a } k\big[q,q^{-1}\big]\text{–basis of} \quad Ker\Big(\epsilon : \widehat{F}_q\big[SL_n\big] \longrightarrow k\big[q,q^{-1}\big] \Big) \,.$$

Now, by definition $I := Ker\Big(\widehat{F}_q\big[SL_n\big] \xrightarrow{\;\epsilon\;} k\big[q,q^{-1}\big] \xrightarrow{\;ev_1\;} k \Big)$, whence $I = Ker(\epsilon) + (q-1) \cdot \widehat{F}_q\big[SL_n\big]$; therefore a $k\big[q,q^{-1}\big]$–basis of I is $\big(M' \setminus \{1\}\big) \cup \big\{ (q-1) \cdot 1 \big\}$, hence

$$(q-1)^{-1} I \quad \text{has } k\big[q,q^{-1}\big]\text{–basis} \quad (q-1)^{-1} \cdot \big(M' \setminus \{1\} \big) \cup \{1\} \,.$$

The upset is that $\widetilde{F}_q\big[SL_n\big] := \sum_{n \geq 0} \Big((q-1)^{-1} I \Big)^n$ is nothing but the unital $k\big[q, q^{-1}\big]$–subalgebra of $F_q\big[SL_n\big]$ generated by

$$\left\{ r_{ij} := \frac{\rho_{ij} - \delta_{ij}}{q-1} \,\middle|\, i, j = 1, \dots, n \right\} .$$

Then one can directly show that this is a Hopf algebra, and that $\widetilde{F}_q\big[SL_n\big] \xrightarrow{q \to 1} U\big(\mathfrak{sl}_n^{\,*}\big)$ as predicted by Theorem 2.1. Details can be found in [Ga2], §§ 2, 4, up to the following changes: the algebra which is considered in [*loc. cit.*] has generators $\big(1+q^{-1}\big)^{-1} r_{ij}$ instead of our r_{ij}, and also generators $r_{ii} := \rho_{ii} = 1 + (q-1)\chi_i$: therefore the presentation given in §2.8 of [*loc. cit.*] has to be changed accordingly; computing the specialization then goes exactly along the same lines, and gives the same result — specialized generators are rescaled, though, compared with the standard ones given in [*loc. cit.*], §1.

We sketch the case of $n = 2$ (see also [FG]). Using notation $\mathrm{a} := \rho_{1,1}$, $\mathrm{b} := \rho_{1,2}$, $\mathrm{c} := \rho_{2,1}$, $\mathrm{d} := \rho_{2,2}$, we have the relations

$$\mathrm{a}\,\mathrm{b} = q\,\mathrm{b}\,\mathrm{a}\,, \qquad \mathrm{a}\,\mathrm{c} = q\,\mathrm{c}\,\mathrm{a}\,, \qquad \mathrm{b}\,\mathrm{d} = q\,\mathrm{d}\,\mathrm{b}\,, \qquad \mathrm{c}\,\mathrm{d} = q\,\mathrm{d}\,\mathrm{c}\,,$$
$$\mathrm{b}\,\mathrm{c} = \mathrm{c}\,\mathrm{b}\,, \qquad \mathrm{a}\,\mathrm{d} - \mathrm{d}\,\mathrm{a} = \big(q - q^{-1}\big)\mathrm{b}\,\mathrm{c}\,, \qquad \mathrm{a}\,\mathrm{d} - q\,\mathrm{b}\,\mathrm{c} = 1$$

holding in $\widehat{F}_q\big[SL_2\big]$ and in $F_q\big[SL_2\big]$, with

$$\Delta(\mathrm{a}) = \mathrm{a} \otimes \mathrm{a} + \mathrm{b} \otimes \mathrm{c},\ \Delta(\mathrm{b}) = \mathrm{a} \otimes \mathrm{b} + \mathrm{b} \otimes \mathrm{d},\ \Delta(\mathrm{c}) = \mathrm{c} \otimes \mathrm{a} + \mathrm{d} \otimes \mathrm{c},\ \Delta(\mathrm{d}) = \mathrm{c} \otimes \mathrm{b} + \mathrm{d} \otimes \mathrm{d}$$
$$\epsilon(\mathrm{a}) = 1, \epsilon(\mathrm{b}) = 0, \epsilon(\mathrm{c}) = 0, \epsilon(\mathrm{d}) = 1, \quad S(\mathrm{a}) = \mathrm{d}, S(\mathrm{b}) = -q^{+1}\mathrm{b}, S(\mathrm{c}) = -q^{-1}\mathrm{c}, S(\mathrm{d}) = \mathrm{a}\,.$$

Then the elements $H_+ := r_{1,1} = \dfrac{\mathrm{a}-1}{q-1}$, $E := r_{1,2} = \dfrac{\mathrm{b}}{q-1}$, $F := r_{2,1} = \dfrac{\mathrm{c}}{q-1}$ and $H_- := r_{2,2} = \dfrac{\mathrm{d}-1}{q-1}$ generate $\widetilde{F}\big[SL_2\big]$: these generators have relations

$$H_+E = qEH_+ + E\,, \quad H_+F = qFH_+ + F\,, \quad EH_- = qH_-E + E\,, \quad FH_- = qH_-F + F\,,$$
$$EF = FE\,, \quad H_+H_- - H_-H_+ = \big(q - q^{-1}\big)EF\,, \quad H_- + H_+ = (q-1)\big(qEF - H_+H_-\big)$$

and Hopf operations given by

$$\Delta(H_+) = H_+ \otimes 1 + 1 \otimes H_+ + (q-1)\big(H_+ \otimes H_+ + E \otimes F\big)\,, \quad \epsilon(H_+) = 0\,, \quad S(H_+) = H_-$$
$$\Delta(E) = E \otimes 1 + 1 \otimes E + (q-1)\big(H_+ \otimes E + E \otimes H_-\big)\,, \qquad \epsilon(E) = 0\,, \qquad S(E) = -q^{+1}E$$
$$\Delta(F) = F \otimes 1 + 1 \otimes F + (q-1)\big(F \otimes H_+ + H_- \otimes F\big)\,, \qquad \epsilon(F) = 0\,, \qquad S(F) = -q^{-1}F$$
$$\Delta(H_-) = H_- \otimes 1 + 1 \otimes H_- + (q-1)\big(H_- \otimes H_- + F \otimes E\big)\,, \quad \epsilon(H_-) = 0\,, \quad S(H_-) = H_+$$

from which one easily checks that $\widetilde{F}_q\big[SL_2\big] \xrightarrow{q \to 1} U(\mathfrak{sl}_2^{\,*})$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$\widetilde{F}_q\big[SL_2\big] \Big/ (q-1)\, \widetilde{F}_q\big[SL_2\big] \xrightarrow{\ \cong\ } U(\mathfrak{sl}_2^{\,*})$$

exists, given by: $H_\pm \mod (q-1) \mapsto \pm \mathrm{h}$, $E \mod (q-1) \mapsto \mathrm{e}$, $F \mod (q-1) \mapsto \mathrm{f}$; that is, $\widetilde{F}_q\big[SL_2\big]$ specializes to $U(\mathfrak{sl}_2{}^*)$ *as a co-Poisson Hopf algebra*, q.e.d.

Finally, the general case of any semisimple group $G = G^\tau$, with the Poisson structure induced from the Lie bialgebra structure of $\mathfrak{g} = \mathfrak{g}^\tau$, can be treated in a different way. Following [Ga1], §§5–6, $F_q[G]$ can be embedded into a (topological) Hopf algebra $U_q(\mathfrak{g}^*) = U_{q,\varphi}^M(\mathfrak{g}^*)$, so that the image of the integer form $\widehat{F}_q[G]$ lies into a suitable (topological) integer form $\mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)$ of $U_q(\mathfrak{g}^*)$. Now, the analysis given in [*loc. cit.*], when carefully read, shows that $\widetilde{F}_q[G] = F_q[G] \cap \mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)^\vee$; moreover, the latter (intersection) algebra "almost" coincides with the integer form $\mathcal{F}_q[G]$ considered in [*loc. cit.*]: in particular, they have the same specialization at $q = 1$. Since in addition $\mathcal{F}_q[G]$ does specialize to $U(\mathfrak{g}^*)$, the same is true for $\widetilde{F}_q[G]$, q.e.d.

3.7 The identity $\widetilde{\widetilde{F}}_q[G] = \widehat{F}_q[G]$. In this section we verify the validity of the second half of part *(b)* of Theorem 2.1 (*"The functors $\big(U_q, \widehat{U}_q\big) \longmapsto \big(U_q, \widetilde{U}_q\big)$ and $\big(F_q, \widehat{F}_q\big) \longmapsto \big(F_q, \widetilde{F}_q\big)$ are inverse of each other"*). Again, we begin with $G = SL_n$.

From $\Delta(\rho_{ij}) = \sum_{k=1}^n \rho_{i,k} \otimes \rho_{k,j}$, we get $\Delta^{(N)}(\rho_{ij}) = \sum_{k_1,\dots,k_{N-1}=1}^n \rho_{i,k_1} \otimes \rho_{k_1,k_2} \otimes \cdots \otimes \rho_{k_{N-1},j}$, by repeated iteration, whence a simple computation yields

$$\delta_N(r_{ij}) = \sum_{k_1,\dots,k_{N-1}=1}^n (q-1)^{-1} \cdot \big((q-1)\, r_{i,k_1} \otimes (q-1)\, r_{k_1,k_2} \otimes \cdots \otimes (q-1)\, r_{k_{N-1},j}\big) \qquad \forall\; i,j$$

so
$$\delta_N\big((q-1) r_{ij}\big) \in (q-1)^N \widetilde{F}_q\big[SL_n\big] \setminus (q-1)^{N+1} \widetilde{F}_q\big[SL_n\big] \qquad \forall\; i,j\,. \tag{3.4}$$

Now, consider again the set

$$M' := \left\{ \prod_{i>j} \rho_{ij}^{N_{ij}} \prod_{h=k} (\rho_{hk}-1)^{N_{hk}} \prod_{l<m} \rho_{lm}^{N_{lm}} \;\middle|\; N_{st} \in \mathbb{N}\; \forall\, s,t\,;\, min\big\{\, N_{1,1}, \dots, N_{n,n} \,\big\} = 0 \right\} :$$

since this is a $k\big[q,q^{-1}\big]$–basis of $\widehat{F}_q\big[SL_n\big]$, we have also that

$$M'' := \left\{ \prod_{i>j} r_{ij}^{N_{ij}} \prod_{h=k} r_{hk}^{N_{hk}} \prod_{l<m} r_{lm}^{N_{lm}} \;\middle|\; N_{st} \in \mathbb{N}\; \forall\, s,t\,;\, min\big\{\, N_{1,1}, \dots, N_{n,n} \,\big\} = 0 \right\}$$

is a $k\big[q,q^{-1}\big]$–basis of $\widetilde{F}_q\big[SL_n\big]$. This and (3.4) above imply that $\widetilde{\widetilde{F}}_q\big[SL_n\big]$ is the unital $k\big[q,q^{-1}\big]$–subalgebra of $F_q\big[SL_n\big]$ generated by $\big\{\, (q-1) r_{ij} \,\big|\, i,j = 1,\dots,n \,\big\}$; since $(q-1) r_{ij} = \rho_{ij} - \delta_{ij}$, the latter algebra does coincide with $\widehat{F}_q\big[SL_n\big]$, as expected.

For the general case of any semisimple group $G = G^\tau$, the result can be obtained again by looking at the immersions $F_q[G] \subseteq U_q(\mathfrak{g}^*)$ and $\widehat{F}_q[G] \subseteq \mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)$, and at the identity

$\widetilde{F}_q[G] = F_q[G] \cap \mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)^\vee$ (cf. §3.5); if we try to compute $\widetilde{\mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)}^\vee$ (remarking that the pair $\Big(k(q) \otimes_{k[q,q^{-1}]} \big(\mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*) \big)^\vee, \big(\mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*) \big)^\vee \Big)$ is a (G)QUEA), we have just to apply much the like methods as for $\widetilde{U}_q(\mathfrak{g})$, thus finding a similar result; then from this and the identity $\widetilde{F}_q[G] = F_q[G] \cap \mathcal{U}_{q,\varphi}^M(\mathfrak{g}^*)^\vee$ we eventually find $\widetilde{\widetilde{F}}_q[G] = \widehat{F}_q[G]$, q.e.d.

§ 4 Second example: the three-dimensional Euclidean group E_2

4.1 The classical setting. Let $G := E_2$, the three-dimensional Euclidean group. Its tangent Lie algebra $\mathfrak{g} = \mathfrak{e}_2$ is generated by f, h, e with relations $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = 0$. The formulae $\delta(f) = h \otimes f - f \otimes h$, $\delta(h) = 0$, $\delta(e) = h \otimes e - e \otimes h$, make $\mathfrak{e}_2$ into a Lie bialgebra, hence E_2 into a Poisson group. These same formulae give also a presentation of the co-Poisson Hopf algebra $U(\mathfrak{e}_2)$ (with the standard Hopf structure).

On the other hand, the function algebra $F\big[E_2\big]$ is the unital associative commutative k–algebra with generators b, $a^{\pm 1}$, c, with Poisson Hopf algebra structure given by

$$\Delta(b) = b \otimes a^{-1} + a \otimes b\,, \qquad \Delta\big(a^{\pm 1}\big) = a^{\pm 1} \otimes a^{\pm 1}\,, \qquad \Delta(c) = c \otimes a + a^{-1} \otimes c$$
$$\epsilon(b) = 0\,, \quad \epsilon\big(a^{\pm 1}\big) = 1\,, \quad \epsilon(c) = 0\,, \qquad S(b) = -b\,, \quad S\big(a^{\pm 1}\big) = a^{\mp 1}\,, \quad S(c) = -c$$
$$\big\{a^{\pm 1}, b\big\} = \pm a^{\pm 1} b\,, \qquad \big\{a^{\pm 1}, c\big\} = \pm a^{\pm 1} c\,, \qquad \{b, c\} = 0$$

E_2 can be realized as $E_2 = \big\{ (x,z,y) \,\big|\, x, y \in k, z \in k \setminus \{0\} \big\}$, with group operation

$$(x_1, z_1, y_1) \cdot (x_2, z_2, y_2) = \big(x_1 z_2^{-1} + z_1 x_2\,,\; z_1 z_2\,,\; y_1 z_2 + z_1^{-1} y_2 \big)\,;$$

in particular the centre of E_2 is simply $Z := \big\{(0,1,0), (0,-1,0)\big\}$, so there is only one other connected Poisson group having $\mathfrak{e}_2$ as Lie bialgebra, namely the adjoint group ${}_aE_2 := E_2 \big/ Z$ (the left subscript a stands for "adjoint"). Then $F\big[{}_aE_2\big]$ coincides with the Poisson Hopf subalgebra of $F\big[{}_aE_2\big]$ spanned by products of an even number of generators, i.e. monomials of even degree: as a unital subalgebra, this is generated by ba, $a^{\pm 2}$, and $a^{-1}c$.

The dual Lie bialgebra $\mathfrak{g}^* = {\mathfrak{e}_2}^*$ is the Lie algebra with generators f, h, e, and relations $[\mathrm{h},\mathrm{e}] = 2\mathrm{e}$, $[\mathrm{h},\mathrm{f}] = 2\mathrm{f}$, $[\mathrm{e},\mathrm{f}] = 0$, with Lie cobracket given by $\delta(\mathrm{f}) = \mathrm{f} \otimes \mathrm{h} - \mathrm{h} \otimes \mathrm{f}$, $\delta(\mathrm{h}) = 0$, $\delta(\mathrm{e}) = \mathrm{h} \otimes \mathrm{e} - \mathrm{e} \otimes \mathrm{h}$ (we choose as generators $\mathrm{f} := f^*$, $\mathrm{h} := 2h^*$, $\mathrm{e} := e^*$, where $\big\{f^*, h^*, e^*\big\}$ is the basis of ${\mathfrak{e}_2}^*$ which is the dual of the basis $\{f,h,e\}$ of $\mathfrak{e}_2$). This again gives a presentation of $U\big({\mathfrak{e}_2}^*\big)$ too. The simply connected algebraic Poisson group with tangent Lie bialgebra ${\mathfrak{e}_2}^*$ can be realized as the group of pairs of matrices

$${}_s{E_2}^* := \left\{ \left(\begin{pmatrix} t^{-1} & 0 \\ r & t \end{pmatrix}, \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix} \right) \,\middle|\, t, s \in k, t \in k \setminus \{0\} \right\};$$

this group has centre $Z := \big\{ (I,I), (-I,-I) \big\}$, so there is only one other (Poisson) group with Lie (bi)algebra $\mathfrak{e}_2{}^*$, namely the adjoint group ${}_aE_2{}^* := {}_sE_2{}^* \Big/ Z$.

Therefore $F\big[{}_sE_2{}^*\big]$ is the unital associative commutative k–algebra with generators x, $z^{\pm 1}$, y, with Poisson Hopf structure given by

$$\begin{gathered}
\Delta(x) = x \otimes z^{-1} + z \otimes x \,, \qquad \Delta\big(z^{\pm 1}\big) = z^{\pm 1} \otimes z^{\pm 1} \,, \qquad \Delta(y) = y \otimes z^{-1} + z \otimes y \\
\epsilon(x) = 0 \,, \;\; \epsilon\big(z^{\pm 1}\big) = 1 \,, \;\; \epsilon(y) = 0 \,, \qquad S(x) = -x \,, \;\; S\big(z^{\pm 1}\big) = z^{\mp 1} \,, \;\; S(y) = -y \\
\{x,y\} = 0 \,, \qquad \big\{ z^{\pm 1}, x \big\} = \pm z^{\pm 1} x \,, \qquad \big\{ z^{\pm 1}, y \big\} = \mp z^{\pm 1} y
\end{gathered}$$

(N.B.: with respect to this presentation, we have $\mathrm{f} = \partial_y\big|_e$, $\mathrm{h} = z\,\partial_z\big|_e$, $\mathrm{e} = \partial_x\big|_e$, where e is the identity element of ${}_aE_2{}^*$). Moreover, $F\big[{}_aE_2{}^*\big]$ can be identified with the Poisson Hopf subalgebra of $F\big[{}_aE_2{}^*\big]$ spanned by products of an even number of generators, i.e. monomials of even degree: this is generated, as a unital subalgebra, by xz, $z^{\pm 2}$, and $z^{-1}y$.

4.2 The (G)QUEA's $\big(U_q^s(\mathfrak{e}_2), \widehat{U}_q^s(\mathfrak{e}_2)\big)$ and $\big(U_q^a(\mathfrak{e}_2), \widehat{U}_q^a(\mathfrak{e}_2)\big)$. We turn now to quantizations: the situation is much similar to the $\mathfrak{sl}_2$ case, so we follow the same pattern, but we stress a bit more the occurrence of different groups sharing the same tangent Lie bialgebra.

Let $U_q(\mathfrak{g}) = U_q^s(\mathfrak{e}_2)$ (where the superscript s stands for "simply connected") be the associative unital $k(q)$–algebra with generators F, $L^{\pm 1}$, E, and relations

$$L L^{-1} = 1 = L^{-1} L \,, \quad L^{\pm 1} F = q^{\mp 1} F L^{\pm 1} \,, \quad L^{\pm 1} E = q^{\pm 1} E L^{\pm 1} \,, \quad E F = F E \,.$$

This is a Hopf algebra, with Hopf structure given by

$$\begin{gathered}
\Delta(F) = F \otimes L^{-2} + 1 \otimes F \,, \qquad \Delta\big(L^{\pm 1}\big) = L^{\pm 1} \otimes L^{\pm 1} \,, \qquad \Delta(E) = E \otimes 1 + L^2 \otimes E \\
\epsilon(F) = 0 \,, \;\; \epsilon\big(L^{\pm 1}\big) = 1 \,, \;\; \epsilon(E) = 0 \,, \qquad S(F) = -F L^2 , \;\; S\big(L^{\pm 1}\big) = L^{\mp 1} , \;\; S(E) = -L^{-2} E \,.
\end{gathered}$$

Then let $\widehat{U}_q^s(\mathfrak{e}_2)$ be the $k\big[q,q^{-1}\big]$–subalgebra of $U_q^s(\mathfrak{e}_2)$ generated by F, $D_\pm := \dfrac{L^{\pm 1} - 1}{q-1}$, E. From the definition of $U_q^s(\mathfrak{e}_2)$ one gets a presentation of $\widehat{U}_q^s(\mathfrak{e}_2)$ as the associative unital algebra with generators F, $D_\pm$, E and relations

$$\begin{gathered}
D_+ E = q E D_+ + E \,, \quad F D_+ = q D_+ F + F \,, \quad E D_- = q D_- E + E \,, \quad D_- F = q F D_- + F \\
E F = F E \,, \qquad D_+ D_- = D_- D_+ \,, \qquad D_+ + D_- + (q-1) D_+ D_- = 0 \,;
\end{gathered}$$

with a Hopf structure given by

$$\begin{gathered}
\Delta(E) = E \otimes 1 + 1 \otimes E + 2(q-1) D_+ \otimes E + (q-1)^2 \cdot D_+^2 \otimes E \\
\Delta(D_\pm) = D_\pm \otimes 1 + 1 \otimes D_\pm + (q-1) \cdot D_\pm \otimes D_\pm \\
\Delta(F) = F \otimes 1 + 1 \otimes F + 2(q-1) F \otimes D_- + (q-1)^2 \cdot F \otimes D_-^2 \\
\epsilon(E) = 0 \,, \qquad S(E) = -E - 2(q-1) D_- E - (q-1)^2 D_-^2 E \\
\epsilon(D_\pm) = 0 \,, \qquad S(D_\pm) = D_\mp \\
\epsilon(F) = 0 \,, \qquad S(F) = -F - 2(q-1) F D_+ - (q-1)^2 F D_+^2 \,.
\end{gathered}$$

The "adjoint version" of $U_q^s(\mathfrak{e}_2)$ is the unital subalgebra $U_q^a(\mathfrak{e}_2)$ generated by F, $K^{\pm 1} := L^{\pm 2}$, E, which is clearly a Hopf subalgebra. It also has a $k\big[q,q^{-1}\big]$–integer form $\widehat{U}_q^a(\mathfrak{e}_2)$, the unital $k\big[q,q^{-1}\big]$–subalgebra generated by F, $H_\pm := \dfrac{K^{\pm 1}-1}{q-1}$, E: this has relations

$$EF = FE\,,\;\; H_+E = q^2EH_+ + (q+1)E\,,\;\; FH_+ = q^2H_+F + (q+1)F\,,\;\; H_+H_- = H_-H_+$$
$$EH_- = q^2H_-E + (q+1)E\,,\;\; H_-F = q^2FH_- + (q+1)F\,,\;\; H_+ + H_- + (q-1)H_+H_- = 0$$

and it is a Hopf subalgebra, with Hopf operations given by

$$\Delta(E) = E\otimes 1 + 1\otimes E + (q-1)\cdot H_+\otimes E\,,\qquad \epsilon(E) = 0\,,\qquad S(E) = -E-(q-1)H_-E$$
$$\Delta(H_\pm) = H_\pm\otimes 1 + 1\otimes H_\pm + (q-1)\cdot H_\pm\otimes H_\pm\,,\qquad \epsilon(H_\pm) = 0\,,\qquad S(H_\pm) = H_\mp$$
$$\Delta(F) = F\otimes 1 + 1\otimes F + (q-1)\cdot F\otimes H_-\,,\qquad \epsilon(F) = 0\,,\qquad S(F) = -F-(q-1)FH_+\,.$$

It is easy to check that $\big(U_q^s(\mathfrak{e}_2), \widehat{U}_q^s(\mathfrak{e}_2)\big)$ is a (G)QUEA, whose semiclassical limit is $U(\mathfrak{e}_2)$: in fact, mapping the generators $F \mod (q-1)$, $D_\pm \mod (q-1)$, $E \mod (q-1)$ respectively to f, $\pm h/2$, $e\in U(\mathfrak{e}_2)$ gives a co-Poisson Hopf algebra isomorphism $\widehat{U}_q^s(\mathfrak{e}_2)\Big/(q-1)\,\widehat{U}_q^s(\mathfrak{e}_2) \xrightarrow{\;\cong\;} U(\mathfrak{e}_2)$. Similarly, $\big(U_q^a(\mathfrak{e}_2), \widehat{U}_q^a(\mathfrak{e}_2)\big)$ is a (G)QUEA too, with semiclassical limit $U(\mathfrak{e}_2)$ again: here a co-Poisson Hopf algebra isomorphism between $\widehat{U}_q^a(\mathfrak{e}_2)\Big/(q-1)\,\widehat{U}_q^a(\mathfrak{e}_2)$ and $U(\mathfrak{e}_2)$ is given by mapping the generators $F \mod (q-1)$, $H_\pm \mod (q-1)$, $E \mod (q-1)$ respectively to f, $\pm h$, $e\in U(\mathfrak{e}_2)$.

4.3 Computation of $\widetilde{U}_q(\mathfrak{e}_2)$ and specialization $\widetilde{U}_q(\mathfrak{e}_2) \xrightarrow{\,q\to 1\,} F\big[{E_2}^*\big]$. This section is devoted to compute $\widetilde{U}_q^s(\mathfrak{e}_2)$ and $\widetilde{U}_q^a(\mathfrak{e}_2)$, and their specialization at $q=1$: everything goes on as in §3.3, so we can be more sketchy. ¿From definitions we have, for any $n\in\mathbb{N}$, $\Delta^n(E) = \sum_{s=1}^n K^{\otimes(s-1)}\otimes E\otimes 1^{\otimes(n-s)}$, so $\delta_n(E) = (K-1)^{\otimes(n-1)}\otimes E = (q-1)^{n-1}\cdot H_+^{\otimes(n-1)}\otimes E$, whence $\delta_n\big((q-1)E\big) \in (q-1)^n\,\widehat{U}_q^a(\mathfrak{e}_2)\setminus(q-1)^{n+1}\,\widehat{U}_q^a(\mathfrak{e}_2)$ thus $(q-1)E\in \widetilde{U}_q^a(\mathfrak{e}_2)$, whereas $E\notin\widetilde{U}_q^a(\mathfrak{e}_2)$. Similarly, we have $(q-1)F$, $(q-1)H_\pm\in\widetilde{U}_q^a(\mathfrak{e}_2)\setminus(q-1)\,\widetilde{U}_q^a(\mathfrak{e}_2)$. Therefore $\widetilde{U}_q^a(\mathfrak{e}_2)$ contains the subalgebra U' generated by $\dot{F} := (q-1)F$, $\dot{H}_\pm := (q-1)H_\pm$, $\dot{E} := (q-1)E$. On the other hand, $\widetilde{U}_q^a(\mathfrak{e}_2)$ is clearly the $k\big[q,q^{-1}\big]$–span of the set $\Big\{ F^aH_+^bH_-^cE^d \,\Big|\, a,b,c,d\in\mathbb{N} \Big\}$: more precisely, the set

$$\Big\{ F^aH_+^bK^{-|b/2|}E^d \,\Big|\, a,b,d\in\mathbb{N} \Big\} = \Big\{ F^aH_+^b\big(1+(q-1)H_-\big)^{|b/2|}E^d \,\Big|\, a,b,d\in\mathbb{N} \Big\}$$

is a $k\big[q,q^{-1}\big]$–basis of $\widetilde{U}_q^a(\mathfrak{e}_2)$; therefore, a straightforward computation shows that any element in $\widetilde{U}_q^a(\mathfrak{e}_2)$ does necessarily lie in U', thus $\widetilde{U}_q^a(\mathfrak{e}_2)$ coincides with U'. Moreover, since $\dot{H}_\pm = K^{\pm 1}-1$, the unital algebra $\widetilde{U}_q^a(\mathfrak{e}_2)$ is generated by $\dot{F}$, $K^{\pm 1}$ and $\dot{E}$ as well.

The previous analysis — *mutatis mutandis* — ensures also that $\widetilde{U}_q^s(\mathfrak{e}_2)$ coincides with the unital $k\big[q,q^{-1}\big]$–subalgebra U'' of $\widehat{U}_q^s(\mathfrak{e}_2)$ generated by $\dot{F} := (q-1)F$, $\dot{D}_\pm := (q-1)D_\pm$,

$\dot{E} := (q-1)E$; in particular, $\widetilde{U}_q^a(\mathfrak{e}_2) \subset \widetilde{U}_q^s(\mathfrak{e}_2)$. Furthermore, as $\dot{D}_\pm = L^{\pm 1} - 1$, the unital algebra $\widetilde{U}_q^s(\mathfrak{e}_2)$ is actually generated by $\dot{F}$, $L^{\pm 1}$ and $\dot{E}$ too. Therefore we can single out the following presentation of $\widetilde{U}_q^s(\mathfrak{e}_2)$: it is the unital associative $k\big[q,q^{-1}\big]$–algebra with generators $\mathcal{F} := L\dot{F}$, $\mathcal{L}^{\pm 1} := L^{\pm 1}$, $\mathcal{E} := \dot{E}L^{-1}$ and relations

$$\mathcal{L}\mathcal{L}^{-1} = 1 = \mathcal{L}^{-1}\mathcal{L}\,, \qquad \mathcal{E}\mathcal{F} = \mathcal{F}\mathcal{E}\,, \qquad \mathcal{L}^{\pm 1}\mathcal{F} = q^{\mp 1}\mathcal{F}\mathcal{L}^{\pm 1}\,, \qquad \mathcal{L}^{\pm 1}\mathcal{E} = q^{\pm 1}\mathcal{E}\mathcal{L}^{\pm 1}$$

with Hopf structure given by

$$\Delta(\mathcal{F}) = \mathcal{F} \otimes \mathcal{L}^{-1} + \mathcal{L} \otimes \mathcal{F}\,, \qquad \Delta\big(\mathcal{L}^{\pm 1}\big) = \mathcal{L}^{\pm 1} \otimes \mathcal{L}^{\pm 1}\,, \qquad \Delta(\mathcal{E}) = \mathcal{E} \otimes \mathcal{L}^{-1} + \mathcal{L} \otimes \mathcal{E}$$
$$\epsilon(\mathcal{F}) = 0\,, \;\; \epsilon\big(\mathcal{L}^{\pm 1}\big) = 1\,, \;\; \epsilon(\mathcal{E}) = 0\,, \qquad S(\mathcal{F}) = -\mathcal{F}\,, \;\; S\big(\mathcal{L}^{\pm 1}\big) = \mathcal{L}^{\mp 1}\,, \;\; S(\mathcal{E}) = -\mathcal{E}\,.$$

As $q \rightarrow 1$, this yields a presentation of the function algebra $F\big[{}_sE_2{}^*\big]$, and the Poisson bracket that $F\big[{}_sE_2{}^*\big]$ earns from this quantization process coincides with the one coming from the Poisson structure on ${}_sE_2{}^*$: namely, there is a Poisson Hopf algebra isomorphism

$$\widetilde{U}_q^s(\mathfrak{e}_2) \Big/ (q-1)\, \widetilde{U}_q^s(\mathfrak{e}_2) \xrightarrow{\;\cong\;} F\big[{}_sE_2{}^*\big]$$

given by $\mathcal{E} \mod (q-1) \mapsto x$, $\mathcal{L}^{\pm 1} \mod (q-1) \mapsto z^{\pm 1}$, $\mathcal{F} \mod (q-1) \mapsto y$. That is, $\widetilde{U}_q^s(\mathfrak{e}_2)$ specializes to $F\big[{}_sE_2{}^*\big]$ *as a Poisson Hopf algebra*, as predicted by Theorem 2.1.

In the "adjoint case", from the definition of U' and the identity $\widetilde{U}_q^a(\mathfrak{e}_2) = U'$ we find that $\widetilde{U}_q^a(\mathfrak{e}_2)$ is the unital associative $k\big[q,q^{-1}\big]$–algebra with generators $\dot{F}$, $K^{\pm 1}$, $\dot{E}$ and relations

$$KK^{-1} = 1 = K^{-1}K\,, \qquad \dot{E}\dot{F} = \dot{F}\dot{E}\,, \qquad K^{\pm 1}\dot{F} = q^{\mp 2}\dot{F}K^{\pm 1}\,, \qquad K^{\pm 1}\dot{E} = q^{\pm 2}\dot{E}K^{\pm 1}$$

with Hopf structure given by

$$\Delta\big(\dot{F}\big) = \dot{F} \otimes K^{-1} + 1 \otimes \dot{F}\,, \qquad \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, \qquad \Delta\big(\dot{E}\big) = \dot{E} \otimes 1 + K \otimes \dot{E}$$
$$\epsilon\big(\dot{F}\big) = 0\,, \; \epsilon\big(K^{\pm 1}\big) = 1\,, \; \epsilon\big(\dot{E}\big) = 0\,, \quad S\big(\dot{F}\big) = -\dot{F}K\,, \; S\big(K^{\pm 1}\big) = K^{\mp 1}, \; S\big(\dot{E}\big) = -K^{-1}\dot{E}\,.$$

The upset is that a Poisson Hopf algebra isomorphism

$$\widetilde{U}_q^a(\mathfrak{e}_2) \Big/ (q-1)\, \widetilde{U}_q^a(\mathfrak{e}_2) \xrightarrow{\;\cong\;} F\big[{}_aE_2{}^*\big] \quad \Big(\subset F\big[{}_sE_2{}^*\big]\Big)$$

exists, given by $\dot{E} \mod (q-1) \mapsto xz$, $K^{\pm 1} \mod (q-1) \mapsto z^{\pm 2}$, $\dot{F} \mod (q-1) \mapsto z^{-1}y$, i.e. $\widetilde{U}_q^a(\mathfrak{e}_2)$ specializes to $F\big[{}_aE_2{}^*\big]$ *as a Poisson Hopf algebra*, according to Theorem 2.1.

4.4 The identity $\widetilde{\widetilde{U}}_q(\mathfrak{e}_2) = \widehat{U}_q(\mathfrak{e}_2)$. The goal of this section is to check that the first half of part *(b)* of Theorem 2.1 (*"The functors* $\big(U_q, \widehat{U}_q\big) \longmapsto \big(U_q, \widetilde{U}_q\big)$ *and* $\big(F_q, \widehat{F}_q\big) \longmapsto \big(F_q, \widetilde{F}_q\big)$ *are inverse of each other"*) is verified. In other words, we have to verify that $\widetilde{\widetilde{U}}_q(\mathfrak{e}_2) = \widehat{U}_q(\mathfrak{e}_2)$, as predicted by Theorem 2.1 *(b)*, both for the simply connected and the adjoint version of the (G)QUEA we are dealing with. First, it is easy to see that $\widetilde{U}_q^s(\mathfrak{e}_2)$ is a free $k\big[q,q^{-1}\big]$–module, with basis $\Big\{ \mathcal{F}^a \mathcal{L}^d \mathcal{E}^c \,\Big|\, a,c \in \mathbb{N}, d \in \mathbb{Z} \Big\}$, hence the set $\mathbb{B} := \Big\{ \mathcal{F}^a \big(\mathcal{L}^{\pm 1} - 1\big)^b \mathcal{E}^c \,\Big|\, a,b,c \in \mathbb{N} \Big\}$, is a $k\big[q,q^{-1}\big]$–basis as well. Second, as $\epsilon(\mathcal{F}) = \epsilon\big(\mathcal{L}^{\pm 1} - 1\big) = \epsilon(\mathcal{E}) = 0$, the ideal $J := Ker\Big(\epsilon : \widetilde{U}_q^s(\mathfrak{e}_2) \rightarrow k\big[q,q^{-1}\big]\Big)$ is the span of $\mathbb{B} \setminus \{1\}$.

Now $I := Ker\Big(\widetilde{U}_q^s(\mathfrak{e}_2) \xrightarrow{\ \epsilon\ } k\big[q,q^{-1}\big] \xrightarrow{\ ev_1\ } k\Big) = J + (q-1)\cdot\widetilde{U}_q^s(\mathfrak{e}_2)$, therefore $\widetilde{\widetilde{U}}_q^s(\mathfrak{e}_2) := \sum_{n\geq 0}\Big((q-1)^{-1}I\Big)^n$ is generated — as a unital $k\big[q,q^{-1}\big]$–subalgebra of $U_q^s(\mathfrak{e}_2)$ — by $(q-1)^{-1}\mathcal{F} = LF$, $(q-1)^{-1}(\mathcal{L}-1) = D_+$, $(q-1)^{-1}\big(\mathcal{L}^{-1}-1\big) = D_-$, $(q-1)^{-1}\mathcal{E} = EL^{-1}$, hence by F, $D_\pm$, E, so it coincides with $\widehat{U}_q^s(\mathfrak{e}_2)$, q.e.d.

The situation is entirely similar for the adjoint case: one simply has to change $\mathcal{F}$, $\mathcal{L}^{\pm 1}$, $\mathcal{E}$ respectively with $\dot{F}$, $K^{\pm 1}$, $\dot{E}$, and $D_\pm$ with $H_\pm$, then everything goes through as above.

4.5 The QFA's $\big(F_q[E_2], \widehat{F}_q[E_2]\big)$ and $\big(F_q[{}_aE_2], \widehat{F}_q[{}_aE_2]\big)$. In this and the following sections we look at Theorem 2.1 starting from QFA's, to get (G)QUEA's out of them. We begin by introducing a QFA for the Euclidean groups E_2 and ${}_aE_2$.

Let $\widehat{F}_q\big[E_2\big]$ be the unital associative $k\big[q,q^{-1}\big]$–algebra with generators $\mathrm{a}^{\pm 1}$, b, c and relations

$$\mathrm{a\,b} = q\,\mathrm{b\,a}\,, \qquad \mathrm{a\,c} = q\,\mathrm{c\,a}\,, \qquad \mathrm{b\,c} = \mathrm{c\,b}$$

endowed with the Hopf algebra structure given by

$$\Delta\big(\mathrm{a}^{\pm 1}\big) = \mathrm{a}^{\pm 1}\otimes\mathrm{a}^{\pm 1}\,, \qquad \Delta(\mathrm{b}) = \mathrm{b}\otimes\mathrm{a}^{-1} + \mathrm{a}\otimes\mathrm{b}\,, \qquad \Delta(\mathrm{c}) = \mathrm{c}\otimes\mathrm{a} + \mathrm{a}^{-1}\otimes\mathrm{c}$$

$$\epsilon\big(\mathrm{a}^{\pm 1}\big) = 1\,,\ \ \epsilon(\mathrm{b}) = 0\,,\ \ \epsilon(\mathrm{c}) = 0\,, \qquad S\big(\mathrm{a}^{\pm 1}\big) = \mathrm{a}^{\mp 1}\,,\ \ S(\mathrm{b}) = -q^{+1}\,\mathrm{b}\,,\ \ S(\mathrm{c}) = -q^{-1}\,\mathrm{c}\,;$$

let also $F_q\big[E_2\big]$ be the $k(q)$–algebra obtained from $\widehat{F}_q\big[E_2\big]$ by scalar extension. Define $\widehat{F}_q\big[{}_aE_2\big]$ as the $k\big[q,q^{-1}\big]$–submodule of $\widehat{F}_q\big[E_2\big]$ spanned by the products of an even number of generators, i.e. monomials of even degree in $\mathrm{a}^{\pm 1}$, b, c: this is a unital subalgebra of $\widehat{F}_q\big[E_2\big]$, generated by $\beta := \mathrm{b\,a}$, $\alpha^{\pm 1} := \mathrm{a}^{\pm 2}$, and $\gamma := \mathrm{a}^{-1}\mathrm{c}$; its scalar extension to $k(q)$ is then a unital $k(q)$–subalgebra of $F_q\big[E_2\big]$, which has a similar description: we denote it by $F_q\big[{}_aE_2\big]$. Both $\Big(F_q\big[E_2\big], \widehat{F}_q\big[E_2\big]\Big)$ and $\Big(F_q\big[{}_aE_2\big], \widehat{F}_q\big[{}_aE_2\big]\Big)$ are QFA's, whose semiclassical limit is $F\big[E_2\big]$ and $F\big[{}_aE_2\big]$ respectively.

4.6 Computation of $\widetilde{F}_q[E_2]$ and $\widetilde{F}_q[{}_aE_2]$ and specializations $\widetilde{F}_q[E_2] \xrightarrow{\ q\to 1\ } U(\mathfrak{g}^*)$ and $\widetilde{F}_q[{}_aE_2] \xrightarrow{\ q\to 1\ } U(\mathfrak{g}^*)$. In this section we go and compute $\widetilde{F}_q[G]$ and its semiclassical limit (i.e. its specialization at $q=1$) for both $G = E_2$ and $G = {}_aE_2$.

First, $\widehat{F}_q\big[E_2\big]$ is free over $k\big[q,q^{-1}\big]$, with basis $\Big\{\mathrm{b}^b\mathrm{a}^a\mathrm{c}^c\,\Big|\,a\in\mathbb{Z}, b,c\in\mathbb{N}\Big\}$, so the set $\mathbb{B}_s := \Big\{\mathrm{b}^b\big(\mathrm{a}^{\pm 1}-1\big)^a\mathrm{c}^c\,\Big|\,a,b,c\in\mathbb{N}\Big\}$ is a $k\big[q,q^{-1}\big]$–basis as well. Second, we have $\epsilon(\mathrm{b}) = \epsilon\big(\mathrm{a}^{\pm 1}-1\big) = \epsilon(\mathrm{c}) = 0$, thus the ideal $J := Ker\Big(\epsilon : \widehat{F}_q\big[E_2\big] \longrightarrow k\big[q,q^{-1}\big]\Big)$ is the span of $\mathbb{B}_s\setminus\{1\}$. Now $I := Ker\Big(\widehat{F}_q\big[E_2\big] \xrightarrow{\ \epsilon\ } k\big[q,q^{-1}\big] \xrightarrow{\ ev_1\ } k\Big) = J + (q-1)\cdot\widehat{F}_q\big[E_2\big]$, thus $\widetilde{F}_q\big[E_2\big] := \sum_{n\geq 0}\Big((q-1)^{-1}I\Big)^n$ turns out to be the unital $k\big[q,q^{-1}\big]$–algebra (subalgebra of $F_q\big[E_2\big]$) with generators $D_\pm := \dfrac{\mathrm{a}^{\pm 1}-1}{q-1}$, $E := \dfrac{\mathrm{b}}{q-1}$, and $F := \dfrac{\mathrm{c}}{q-1}$ and relations

$$D_+E = qED_+ + E\,, \quad D_+F = qFD_+ + F\,, \quad ED_- = qD_-E + E\,, \quad FD_- = qD_-F + F$$

$$EF = FE\,, \qquad D_+D_- = D_-D_+\,, \qquad D_+ + D_- + (q-1)D_+D_- = 0\,;$$

with a Hopf structure given by

$$\begin{aligned} \Delta(E) &= E \otimes 1 + 1 \otimes E + (q-1)\big(E \otimes D_- + D_+ \otimes E\big)\,, && \epsilon(E) = 0\,, && S(E) = -q^{+1} E \\ \Delta(D_\pm) &= D_\pm \otimes 1 + 1 \otimes D_\pm + (q-1) \cdot D_\pm \otimes D_\pm\,, && \epsilon(D_\pm) = 0\,, && S(D_\pm) = D_\mp \\ \Delta(F) &= F \otimes 1 + 1 \otimes F + (q-1)\big(F \otimes D_+ + D_- \otimes F\big)\,, && \epsilon(F) = 0\,, && S(F) = -q^{-1} F\,. \end{aligned}$$

This implies that $\widetilde{F}_q\big[E_2\big] \xrightarrow{\;q \rightarrow 1\;} U(\mathfrak{e}_2{}^*)$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$\widetilde{F}_q\big[E_2\big] \Big/ (q-1)\, \widetilde{F}_q\big[E_2\big] \xrightarrow{\;\cong\;} U(\mathfrak{e}_2{}^*)$$

exists, given by $D_\pm \mod (q-1) \mapsto \pm \mathrm{h}/2\,,\; E \mod (q-1) \mapsto \mathrm{e}\,,\; F \mod (q-1) \mapsto \mathrm{f}\,$; thus $\widetilde{F}_q\big[E_2\big]$ does specialize to $U(\mathfrak{e}_2{}^*)$ *as a co-Poisson Hopf algebra*, q.e.d.

Similarly, if we consider $\widehat{F}_q\big[{}_aE_2\big]$ the same analysis works again. In fact, $\widehat{F}_q\big[{}_aE_2\big]$ is free over $k\big[q,q^{-1}\big]$, with basis $\mathbb{B}_a := \Big\{ \beta^b \big(\alpha^{\pm 1} - 1\big)^a \gamma^c \,\Big|\, a,b,c \in \mathbb{N} \Big\}$; therefore, as above the ideal $J := Ker\Big(\epsilon : \widehat{F}_q\big[{}_aE_2\big] \rightarrow k\big[q,q^{-1}\big]\Big)$ is the span of $\mathbb{B}_a \setminus \{1\}$. Now, we have $I := Ker\Big(\widehat{F}_q\big[{}_aE_2\big] \overset{\epsilon}{\twoheadrightarrow} k\big[q,q^{-1}\big] \overset{ev_1}{\twoheadrightarrow} k\Big) = J + (q-1) \cdot \widehat{F}_q\big[{}_aE_2\big]$, so $\widetilde{F}_q\big[{}_aE_2\big] := \sum_{n \geq 0} \Big((q-1)^{-1} I\Big)^n$ is nothing but the unital $k\big[q,q^{-1}\big]$–algebra (subalgebra of $F_q\big[{}_aE_2\big]$) with generators $H_\pm := \dfrac{\alpha^{\pm 1} - 1}{q-1}$, $E' := \dfrac{\beta}{q-1}$, and $F' := \dfrac{\gamma}{q-1}$ and relations

$$\begin{gathered} E'F' = q^{-2} F'E',\; H_+E' = q^2 E'H_+ + (q+1)E',\; H_+F' = q^2 F'H_+ + (q+1)F',\; H_+H_- = H_-H_+ \\ E'H_- = q^2 H_-E' + (q+1)E',\; F'H_- = q^2 H_-F' + (q+1)F',\; H_+ + H_- + (q-1)H_+H_- = 0 \end{gathered}$$

with a Hopf structure given by

$$\begin{aligned} \Delta(E') &= E' \otimes 1 + 1 \otimes E' + (q-1) \cdot H_+ \otimes E'\,, && \epsilon(E') = 0\,, && S(E') = -E' - (q-1) E'H_- \\ \Delta(H_\pm) &= H_\pm \otimes 1 + 1 \otimes H_\pm + (q-1) \cdot H_\pm \otimes H_\pm\,, && \epsilon(H_\pm) = 0\,, && S(H_\pm) = H_\mp \\ \Delta(F') &= F' \otimes 1 + 1 \otimes F' + (q-1) \cdot H_- \otimes F'\,, && \epsilon(F') = 0\,, && S(F') = -F' - (q-1) F'H_+\,. \end{aligned}$$

This implies that $\widetilde{F}_q\big[{}_aE_2\big] \xrightarrow{\;q \rightarrow 1\;} U(\mathfrak{e}_2{}^*)$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$\widetilde{F}_q\big[{}_aE_2\big] \Big/ (q-1)\, \widetilde{F}_q\big[{}_aE_2\big] \xrightarrow{\;\cong\;} U(\mathfrak{e}_2{}^*)$$

is given by $H_\pm \mod (q-1) \mapsto \pm \mathrm{h}\,,\; E' \mod (q-1) \mapsto \mathrm{e}\,,\; F' \mod (q-1) \mapsto \mathrm{f}\,$; so $\widetilde{F}_q\big[{}_aE_2\big]$ too specializes to $U(\mathfrak{e}_2{}^*)$ *as a co-Poisson Hopf algebra*, as expected.

4.7 The identities $\widehat{\widetilde{F}}_q[E_2] = \widehat{F}_q[E_2]$ and $\widehat{\widetilde{F}}_q[{}_aE_2] = \widehat{F}_q[{}_aE_2]$. In this section we verify the validity of the second half of part *(b)* of Theorem 2.1 (*"The functors $\big(U_q, \widehat{U}_q\big) \longmapsto \big(U_q, \widetilde{U}_q\big)$ and $\big(F_q, \widehat{F}_q\big) \longmapsto \big(F_q, \widetilde{F}_q\big)$ are inverse of each other"*). In other words, we check that $\widehat{\widetilde{F}}_q[E_2] = \widehat{F}_q[E_2]$ and $\widehat{\widetilde{F}}_q[{}_aE_2] = \widehat{F}_q[{}_aE_2]$.

By induction we find formulae $\Delta^n(E) = \sum_{r+s+1=n} \mathrm{a}^{\otimes r} \otimes E \otimes \big(\mathrm{a}^{-1}\big)^{\otimes s}$, $\Delta^n(D_\pm) = \sum_{r+s+1=n} \big(\mathrm{a}^{\pm 1}\big)^{\otimes r} \otimes D_\pm \otimes 1^{\otimes s}$, and $\Delta^n(F) = \sum_{r+s+1=n} \big(\mathrm{a}^{-1}\big)^{\otimes r} \otimes E \otimes \mathrm{a}^{\otimes s}$: these imply

$$\delta_n(E) = \sum_{r+s+1=n} (\mathrm{a}-1)^{\otimes r} \otimes E \otimes \big(\mathrm{a}^{-1}-1\big)^{\otimes s} = (q-1)^{n-1} \sum_{r+s+1=n} {D_+}^{\otimes r} \otimes E \otimes {D_-}^{\otimes s}$$
$$\delta_n(D_\pm) = \big(\mathrm{a}^{\pm 1}-1\big)^{\otimes (n-1)} \otimes D_\pm = (q-1)^{n-1} {D_\pm}^{\otimes n}$$
$$\delta_n(F) = \sum_{r+s+1=n} \big(\mathrm{a}^{-1}-1\big)^{\otimes r} \otimes E \otimes (\mathrm{a}-1)^{\otimes s} = (q-1)^{n-1} \sum_{r+s+1=n} {D_-}^{\otimes r} \otimes E \otimes {D_+}^{\otimes s}$$

which gives $\dot{E} := (q-1)E$, $\dot{D}_\pm := (q-1)D_\pm$, $\dot{F} := (q-1)F \in \widetilde{\widetilde{F}}_q\big[E_2\big] \setminus (q-1)\widetilde{\widetilde{F}}_q\big[E_2\big]$. So $\widetilde{\widetilde{F}}_q\big[E_2\big]$ contains the unital $k\big[q,q^{-1}\big]$–subalgebra A' generated (inside $F_q[E_2]$) by $\dot{E}$, $\dot{D}_\pm$ and $\dot{F}$; but $\dot{E} = \mathrm{b}$, $\dot{D}_\pm = \mathrm{a}^{\pm 1} - 1$, and $\dot{F} = \mathrm{c}$, thus A' is just $\widehat{F}_q[E_2]$. Since $\widetilde{F}_q[E_2]$ is the $k\big[q,q^{-1}\big]$–span of the set $\Big\{ E^e D_+^{d_+} D_-^{d_-} F^f \,\Big|\, e, d_+, d_-, f \in \mathbb{N} \Big\}$, one easily sees — using the previous formulae for Δ^n — that in fact $\widetilde{\widetilde{F}}_q[E_2]$ must coincide with $A' = \widehat{F}_q[E_2]$, q.e.d.

When dealing with the adjoint case, the previous arguments go through again: in fact, $\widetilde{\widetilde{F}}_q\big[{}_aE_2\big]$ turns out to coincide with the unital $k\big[q,q^{-1}\big]$–subalgebra A'' generated (inside $F_q\big[{}_aE_2\big]$) by $\dot{E}' := (q-1)E' = \beta$, $\dot{H}_\pm := (q-1)H_\pm = \alpha^{\pm 1} - 1$, and $\dot{F}' := (q-1)F' = \gamma$; but this is also generated by β, $\alpha^{\pm 1}$ and γ, thus it coincides with $\widehat{F}_q\big[{}_aE_2\big]$, q.e.d.

§ 5 Third example: the Heisenberg group H_n

5.1 The classical setting. Let $G := H_n$, the $(2n+1)$–dimensional Heisenberg group. Its tangent Lie algebra $\mathfrak{g} = \mathfrak{h}_n$ is generated by $\{ f_i, h, e_i \,|\, i = 1, \dots, n \}$ with relations $[e_i, f_j] = \delta_{ij} h$, $[e_i, e_j] = [f_i, f_j] = [h, e_i] = [h, f_j] = 0$ $(\forall\, i, j = 1, \dots n)$. The formulae $\delta(f_i) = f_i \otimes h - h \otimes f_i$, $\delta(h) = 0$, $\delta(e_i) = e_i \otimes h - h \otimes e_i$ $(\forall\, i = 1, \dots n)$ make $\mathfrak{h}_n$ into a Lie bialgebra, which provides H_n with a structure of Poisson group; these same formulae give also a presentation of the co-Poisson Hopf algebra $U(\mathfrak{h}_n)$ (with the standard Hopf structure). The group H_n is usually defined as the group of all square matrices $\big(a_{ij}\big)_{i,j=1,\dots,n+2;}$ such that $a_{ii} = 1$ $\forall\, i$ and $a_{ij} = 0$ $\forall\, i, j$ such that either $i > j$ or $1 \neq i < j$ or $i < j \neq n+2$; it can also be realized as $H_n = k^n \times k \times k^n$ with group operation given by $\big(\underline{x}', z', \underline{y}'\big) \cdot \big(\underline{x}'', z'', \underline{y}''\big) = \big(\underline{x}' + \underline{x}'', z' + z'' + \underline{x}' * \underline{y}'', \underline{y}' + \underline{y}''\big)$, where we use vector notation $\underline{v} = (v_1, \dots, v_n) \in k^n$ and $\underline{x}' * \underline{y}'' := \sum_{i=1}^n x'_i y''_i$ is the standard scalar product in k^n; in particular the identity of H_n is $e = (\underline{0}, 0, \underline{0})$ and the inverse of a generic element is given by $\big(\underline{x}, z, \underline{y}\big)^{-1} = \big(-\underline{x}\,, -z + \underline{x} * \underline{y}\,, -\underline{y}\big)$. Therefore, the function algebra $F\big[H_n\big]$ is the unital associative commutative k–algebra with generators a_1, …, a_n, c, b_1, …, b_n, and with Poisson Hopf algebra structure given by

$$\Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i\,, \;\; \Delta(c) = c \otimes 1 + 1 \otimes c + \sum_{\ell=1}^{n} a_\ell \otimes b_\ell\,, \;\; \Delta(b_i) = b_i \otimes 1 + 1 \otimes b_i$$

$$\epsilon(a_i) = 0\,, \;\; \epsilon(c) = 0\,, \;\; \epsilon(b_i) = 0\,, \qquad S(a_i) = -a_i\,, \;\; S(c) = -c + \sum_{\ell=1}^{n} a_\ell b_\ell\,, \;\; S(b_i) = -b_i$$

$$\{a_i, a_j\} = 0\,, \qquad \{a_i, b_j\} = 0\,, \qquad \{b_i, b_j\} = 0\,, \qquad \{a_i, c\} = a_i\,, \qquad \{b_i, c\} = b_i$$

for all $i, j = 1, \dots, n$. (N.B.: with respect to this presentation, we have $f_i = \partial_{b_i}\big|_e$, $h = \partial_c\big|_e$, $e_i = \partial_{a_i}\big|_e$, where e is the identity element of H_n). The dual Lie bialgebra $\mathfrak{g}^* = {\mathfrak{h}_n}^*$ is the Lie algebra with generators f_i, h, e_i, and relations $[\mathrm{e}_i, \mathrm{h}] = \mathrm{e}_i$, $[\mathrm{f}_i, \mathrm{h}] = \mathrm{f}_i$, $[\mathrm{e}_i, \mathrm{e}_j] = [\mathrm{e}_i, \mathrm{f}_j] = [\mathrm{f}_i, \mathrm{f}_j] = 0$, with Lie cobracket given by $\delta(\mathrm{f}_i) = 0$, $\delta(\mathrm{h}) = \sum_{j=1}^{n} (\mathrm{e}_j \otimes \mathrm{f}_j - \mathrm{f}_j \otimes \mathrm{e}_j)$, $\delta(\mathrm{e}_i) = 0$ for all $i = 1, \dots, n$ (we take $\mathrm{f}_i := f_i^*$, $\mathrm{h} := h^*$, $\mathrm{e}_i := e_i^*$, where $\big\{ f_i^*, h^*, e_i^* \,\big|\, i = 1, \dots, n \big\}$ is the basis of ${\mathfrak{h}_n}^*$ which is the dual of the basis $\{ f_i, h, e_i \,|\, i = 1, \dots, n \}$ of $\mathfrak{h}_n$). This again gives a presentation of $U({\mathfrak{h}_n}^*)$ too. The simply connected algebraic Poisson group with tangent Lie bialgebra ${\mathfrak{h}_n}^*$ can be realized (with $k^\star := k \setminus \{0\}$) as ${}_sH_n{}^* = k^n \times k^\star \times k^n$, with group operation given by $\big(\underline{\dot{x}}, \dot{z}, \underline{\dot{y}}\big) \cdot \big(\underline{\check{x}}, \check{z}, \underline{\check{y}}\big) = \big(\check{z}\underline{\dot{x}} + \dot{z}^{-1}\underline{\check{x}}, \dot{z}\check{z}, \check{z}\underline{\dot{y}} + \dot{z}^{-1}\underline{\check{y}}\big)$; so the identity of ${}_sH_n{}^*$ is $e = (\underline{0}, 1, \underline{0})$ and the inverse is given by $\big(\underline{x}, z, \underline{y}\big)^{-1} = \big(-\underline{x}, z^{-1}, -\underline{y}\big)$. Its centre is $Z\big({}_sH_n{}^*\big) = \big\{(\underline{0}, 1, \underline{0}), (\underline{0}, -1, \underline{0})\big\} =: Z$, so there is only one other (Poisson) group with tangent Lie bialgebra ${\mathfrak{h}_n}^*$, that is the adjoint group ${}_aH_n{}^* := {}_sH_n{}^* \big/ Z$.

It is clear that $F\big[{}_sH_n{}^*\big]$ is the unital associative commutative k–algebra with generators $\alpha_1, \dots, \alpha_n$, $\gamma^{\pm 1}$, $\beta_1, \dots, \beta_n$, and with Poisson Hopf algebra structure given by

$$\Delta(\alpha_i) = \alpha_i \otimes \gamma + \gamma^{-1} \otimes \alpha_i\,, \qquad \Delta\big(\gamma^{\pm 1}\big) = \gamma^{\pm 1} \otimes \gamma^{\pm 1}, \qquad \Delta(\beta_i) = \beta_i \otimes \gamma + \gamma^{-1} \otimes \beta_i$$

$$\epsilon(\alpha_i) = 0\,, \;\; \epsilon\big(\gamma^{\pm 1}\big) = 1\,, \;\; \epsilon(\beta_i) = 0\,, \qquad S(\alpha_i) = -\alpha_i\,, \;\; S\big(\gamma^{\pm 1}\big) = \gamma^{\mp 1}, \;\; S(\beta_i) = -\beta_i$$

$$\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta_j\} = \{\beta_i, \beta_j\} = \{\alpha_i, \gamma\} = \{\beta_i, \gamma\} = 0\,, \qquad \{\alpha_i, \beta_j\} = \delta_{ij} \big(\gamma^2 - \gamma^{-2}\big) \big/ 2$$

for all $i, j = 1, \dots, n$ (N.B.: with respect to this presentation, we have $\mathrm{f}_i = \partial_{\beta_i}\big|_e$, $\mathrm{h} = \frac{1}{2}\, \gamma\, \partial_\gamma\big|_e$, $\mathrm{e}_i = \partial_{\alpha_i}\big|_e$, where e is the identity element of ${}_sH_n{}^*$), and $F\big[{}_aH_n{}^*\big]$ can be identified — as in the case of the Euclidean group — with the Poisson Hopf subalgebra of $F\big[{}_aH_n{}^*\big]$ which is spanned by products of an even number of generators: this is generated, as a unital subalgebra, by $\alpha_i\gamma$, $\gamma^{\pm 2}$, and $\gamma^{-1}\beta_i$ ($i = 1, \dots, n$).

5.2 The (G)QUEA's $\big(U_q^s(\mathfrak{h}_n), \widehat{U}_q^s(\mathfrak{h}_n)\big)$ and $\big(U_q^a(\mathfrak{h}_n), \widehat{U}_q^a(\mathfrak{h}_n)\big)$. We switch now to quantizations, following again the pattern of $\mathfrak{sl}_2$. Let $U_q(\mathfrak{g}) = U_q^s(\mathfrak{h}_n)$ be the associative unital $k(q)$–algebra with generators F_i, $L^{\pm 1}$, E_i ($i = 1, \dots, n$) and relations

$$L L^{-1} = 1 = L^{-1} L\,, \quad L^{\pm 1} F = F L^{\pm 1}\,, \quad L^{\pm 1} E = E L^{\pm 1}\,, \quad E_i F_j - F_j E_i = \delta_{ij} \frac{L - L^{-2}}{q - q^{-1}}$$

for all $i, j = 1, \dots, n$; we give it a structure of Hopf algebra, by setting ($\forall\, i, j = 1, \dots, n$)

$$\Delta(E_i) = E_i \otimes L^2 + 1 \otimes E_i\,, \quad \Delta\big(L^{\pm 1}\big) = L^{\pm 1} \otimes L^{\pm 1}, \quad \Delta(F_i) = F_i \otimes 1 + L^{-2} \otimes F_i$$

$$\epsilon(E_i) = 0\,, \;\; \epsilon\big(L^{\pm 1}\big) = 1\,, \;\; \epsilon(F_i) = 0\,, \quad S(E_i) = -E_i L^{-2}, \;\; S\big(L^{\pm 1}\big) = L^{\mp 1}, \;\; S(F_i) = -L^2 F_i$$

Note also that $\left\{ \prod_{i=1}^n F_i^{a_i} \cdot L^z \cdot \prod_{i=1}^n E_i^{d_i} \,\middle|\, z \in \mathbb{Z},\, a_i, d_i \in \mathbb{N},\, \forall\, i \right\}$ is a $k(q)$–basis of $U_q^s(\mathfrak{h}_n)$.

Now let $\widehat{U}_q^s(\mathfrak{h}_n)$ be the $k\big[q,q^{-1}\big]$–subalgebra of $U_q^s(\mathfrak{h}_n)$ generated by $F_1, \dots, F_n$, $D := \dfrac{L-1}{q-1}$, $\varGamma := \dfrac{L-L^{-2}}{q-q^{-1}}$, $E_1, \dots, E_n$. Then $\widehat{U}_q^s(\mathfrak{h}_n)$ can be presented as the associative unital algebra with generators $F_1, \dots, F_n, L^{\pm 1}, D, \varGamma, E_1, \dots, E_n$ and relations

$$\begin{gathered} DX = XD\,, \qquad L^{\pm 1} X = X L^{\pm 1}\,, \qquad \varGamma X = X \varGamma\,, \qquad E_i F_j - F_j E_i = \delta_{ij} \varGamma \\ L = 1 + (q-1) D\,, \qquad L^2 - L^{-2} = \big(q - q^{-1}\big) \varGamma\,, \qquad D(L+1)\big(1+L^{-2}\big) = \big(1+q^{-1}\big) \varGamma \end{gathered}$$

for all $X \in \big\{F_i, L^{\pm 1}, D, \varGamma, E_i\big\}_{i=1,\dots,n}$ and $i,j = 1, \dots, n$; furthermore, $\widehat{U}_q^s(\mathfrak{h}_n)$ is a Hopf subalgebra (over $k\big[q,q^{-1}\big]$), with

$$\begin{array}{ccc} \Delta(\varGamma) = \varGamma \otimes L^2 + L^{-2} \otimes \varGamma\,, & \epsilon(\varGamma) = 0\,, & S(\varGamma) = -\varGamma \\ \Delta(D) = D \otimes 1 + L \otimes D\,, & \epsilon(D) = 0\,, & S(D) = -L^{-1} D\,. \end{array}$$

Moreover, from relations $L = 1 + (q-1) D$ and $L^{-1} = L^3 - \big(q - q^{-1}\big) L \varGamma$ it follows that

$$\widehat{U}_q^s(\mathfrak{h}_n) = \; k\big[q,q^{-1}\big]\text{–span of} \left\{ \prod_{i=1}^n F_i^{a_i} \cdot D^b \varGamma^c \cdot \prod_{i=1}^n E_i^{d_i} \,\middle|\, a_i, b, c, d_i \in \mathbb{N},\, \forall\, i = 1, \dots, n \right\} \tag{5.1}$$

The "adjoint version" of $U_q^s(\mathfrak{h}_n)$ is the unital subalgebra $U_q^a(\mathfrak{h}_n)$ generated by F_i, $K^{\pm 1} := L^{\pm 2}$, E_i $(i = 1, \dots, n)$, which is clearly a Hopf subalgebra. It also has a $k\big[q,q^{-1}\big]$–integer form $\widehat{U}_q^a(\mathfrak{e}_2)$, namely the unital $k\big[q,q^{-1}\big]$–subalgebra generated by $F_1, \dots, F_n$, $K^{\pm 1}$, $H := \dfrac{K-1}{q-1}$, $\varGamma := \dfrac{K - K^{-1}}{q - q^{-1}}$, $E_1, \dots, E_n$: this has relations

$$\begin{gathered} HX = XH\,, \qquad K^{\pm 1} X = X K^{\pm 1}\,, \qquad \varGamma X = X \varGamma\,, \qquad E_i F_j - F_j E_i = \delta_{ij} \varGamma \\ K = 1 + (q-1) H\,, \qquad K - K^{-1} = \big(q - q^{-1}\big) \varGamma\,, \qquad H\big(1 + K^{-1}\big) = \big(1 + q^{-1}\big) \varGamma \end{gathered}$$

for all $X \in \big\{F_i, K^{\pm 1}, H, \varGamma, E_i\big\}_{i=1,\dots,n}$ and $i,j = 1, \dots, n$, and Hopf operations given by

$$\begin{array}{ccc} \Delta(E_i) = E_i \otimes K + 1 \otimes E_i\,, & \epsilon(E_i) = 0\,, & S(E_i) = -E_i K^{-1} \\ \Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, & \epsilon\big(K^{\pm 1}\big) = 1\,, & S\big(K^{\pm 1}\big) = K^{\mp 1} \\ \Delta(H) = H \otimes 1 + K \otimes H\,, & \epsilon(H) = 0\,, & S(H) = -K^{-1} H \\ \Delta(\varGamma) = \varGamma \otimes K^{-1} + K \otimes \varGamma\,, & \epsilon(\varGamma) = 0\,, & S(\varGamma) = -\varGamma \\ \Delta(F_i) = F_i \otimes 1 + K^{-1} \otimes F_i\,, & \epsilon(F_i) = 0\,, & S(F_i) = -K^{+1} F_i \end{array}$$

for all $i = 1, \dots, n$. One can easily check that $\big(U_q^s(\mathfrak{h}_n), \widehat{U}_q^s(\mathfrak{h}_n)\big)$ is a (G)QUEA, with $U(\mathfrak{h}_n)$ as semiclassical limit: in fact, mapping the generators $F_i \mod (q-1)$, $L^{\pm 1} \mod (q-1)$, $D \mod (q-1)$, $\varGamma \mod (q-1)$, $E_i \mod (q-1)$ respectively to $f_i, 1, h/2, h$, $e_i \in U(\mathfrak{h}_n)$ yields a co-Poisson Hopf algebra isomorphism between $\widehat{U}_q^s(\mathfrak{h}_n) \Big/ (q-1)\, \widehat{U}_q^s(\mathfrak{h}_n)$

and $U(\mathfrak{h}_n)$. Similarly, $\big(U_q^a(\mathfrak{h}_n), \widehat{U}_q^a(\mathfrak{h}_n)\big)$ is a (G)QUEA too, again with limit $U(\mathfrak{h}_n)$, for a co-Poisson Hopf algebra isomorphism between $\widehat{U}_q^a(\mathfrak{h}_n) \Big/ (q-1)\,\widehat{U}_q^a(\mathfrak{h}_n)$ and $U(\mathfrak{h}_n)$ is given by mapping the generators $F_i \mod (q-1)$, $K^{\pm 1} \mod (q-1)$, $H \mod (q-1)$, $\varGamma \mod (q-1)$, $E_i \mod (q-1)$ respectively to f_i, 1, h, h, $e_i \in U(\mathfrak{h}_n)$.

5.3 Computation of $\widetilde{U}_q(\mathfrak{h}_n)$ and specialization $\widetilde{U}_q(\mathfrak{h}_n) \xrightarrow{\;q\to 1\;} F\big[{H_n}^*\big]$. Here we compute $\widetilde{U}_q^s(\mathfrak{h}_n)$ and $\widetilde{U}_q^a(\mathfrak{h}_n)$, and their semiclassical limits, again along the pattern of §3.3.

Definitions give, for any $n \in \mathbb{N}$, $\Delta^n(E_i) = \sum_{s=1}^n 1^{\otimes (s-1)} \otimes E_i \otimes (L^2)^{\otimes (n-s)}$, hence $\delta_n(E_i) = (q-1)^{n-1} \cdot E_i \otimes D^{\otimes (n-1)}$ so $\delta_n\big((q-1)E\big) \in (q-1)^n\, \widehat{U}_q^s(\mathfrak{h}_n) \setminus (q-1)^{n+1}\, \widehat{U}_q^s(\mathfrak{h}_n)$ whence $\dot{E}_i := (q-1)E_i \in \widetilde{U}_q^s(\mathfrak{e}_2)$, whereas $E_i \notin \widetilde{U}_q^s(\mathfrak{h}_n)$; similarly, we have $\dot{F}_i := (q-1)F_i$, $L^{\pm 1}$, $\dot{D} := (q-1)D = L-1$, $\dot{\varGamma} := (q-1)\varGamma \in \widetilde{U}_q^s(\mathfrak{h}_n) \setminus (q-1)\, \widetilde{U}_q^s(\mathfrak{h}_n)$, for all $i = 1, \dots, n$. Therefore $\widetilde{U}_q^s(\mathfrak{h}_n)$ contains the subalgebra U' generated by $\dot{F}_i$, $L^{\pm 1}$, $\dot{D}$, $\dot{\varGamma}$, $\dot{E}_i$; the upset is that in fact $\widetilde{U}_q^s(\mathfrak{h}_n) = U'$: this is easily seen — acting like in the case of SL_2 and of E_2 — using the formulae above along with (3.4). As a consequence, $\widetilde{U}_q^s(\mathfrak{h}_n)$ is the unital $k\big[q,q^{-1}\big]$–algebra with generators $\dot{F}_1, \dots, \dot{F}_n$, $L^{\pm 1}$, $\dot{D}$, $\dot{\varGamma}$, $\dot{E}_1, \dots, \dot{E}_n$ and relations

$$\dot{D}\dot{X} = \dot{X}\dot{D}\,, \qquad L^{\pm 1}\dot{X} = \dot{X}L^{\pm 1}\,, \qquad \dot{\varGamma}\dot{X} = \dot{X}\dot{\varGamma}\,, \qquad \dot{E}_i\dot{F}_j - \dot{F}_j\dot{E}_i = \delta_{ij}(q-1)\dot{\varGamma}$$
$$L = 1 + \dot{D}\,, \qquad L^2 - L^{-2} = \big(1+q^{-1}\big)\dot{\varGamma}\,, \qquad \dot{D}(L+1)\big(1+L^{-2}\big) = \big(1+q^{-1}\big)\dot{\varGamma}$$

for all $\dot{X} \in \big\{\dot{F}_i, L^{\pm 1}, \dot{D}, \dot{\varGamma}, \dot{E}_i\big\}_{i=1,\dots,n}$ and $i, j = 1, \dots, n$, with Hopf structure given by

$$\begin{array}{cccc}
\Delta\big(\dot{E}_i\big) = \dot{E}_i \otimes L^2 + 1 \otimes \dot{E}_i\,, & \epsilon\big(\dot{E}_i\big) = 0\,, & S\big(\dot{E}_i\big) = -\dot{E}_i L^{-2} & \forall\, i = 1, \dots, n \\
\Delta\big(L^{\pm 1}\big) = L^{\pm 1} \otimes L^{\pm 1}\,, & \epsilon\big(L^{\pm 1}\big) = 1\,, & S\big(L^{\pm 1}\big) = L^{\mp 1} & \\
\Delta\big(\dot{\varGamma}\big) = \dot{\varGamma} \otimes L^2 + L^{-2} \otimes \dot{\varGamma}\,, & \epsilon\big(\dot{\varGamma}\big) = 0\,, & S\big(\dot{\varGamma}\big) = -\varGamma & \\
\Delta\big(\dot{D}\big) = \dot{D} \otimes 1 + L \otimes \dot{D}\,, & \epsilon\big(\dot{D}\big) = 0\,, & S\big(\dot{D}\big) = -L^{-1}\dot{D} & \\
\Delta\big(\dot{F}_i\big) = \dot{F}_i \otimes 1 + L^{-2} \otimes \dot{F}_i\,, & \epsilon\big(F_i\big) = 0\,, & S\big(\dot{F}_i\big) = -L^2\dot{F}_i & \forall\, i = 1, \dots, n\,.
\end{array}$$

A similar analysis shows that $\widetilde{U}_q^a(\mathfrak{h}_n)$ coincides with the unital $k\big[q,q^{-1}\big]$–subalgebra U'' of $\widehat{U}_q^a(\mathfrak{h}_n)$ generated by $\dot{F}_i$, $K^{\pm 1}$, $\dot{H} := (q-1)H$, $\dot{\varGamma}$, $\dot{E}_i$ $(i = 1, \dots, n)$; in particular, $\widetilde{U}_q^a(\mathfrak{h}_n) \subset \widetilde{U}_q^s(\mathfrak{h}_n)$. Therefore $\widetilde{U}_q^a(\mathfrak{h}_n)$ can be presented as the unital associative $k\big[q,q^{-1}\big]$–algebra with generators $\dot{F}_1, \dots, \dot{F}_n$, $\dot{H}$, $K^{\pm 1}$, $\dot{\varGamma}$, $\dot{E}_1, \dots, \dot{E}_n$ and relations

$$\dot{H}\dot{X} = \dot{X}\dot{H}\,, \qquad K^{\pm 1}\dot{X} = \dot{X}K^{\pm 1}\,, \qquad \dot{\varGamma}\dot{X} = \dot{X}\dot{\varGamma}\,, \qquad \dot{E}_i\dot{F}_j - \dot{F}_j\dot{E}_i = \delta_{ij}(q-1)\dot{\varGamma}$$
$$K = 1 + \dot{H}\,, \qquad K - K^{-1} = \big(1+q^{-1}\big)\dot{\varGamma}\,, \qquad \dot{H}\big(1+K^{-1}\big) = \big(1+q^{-1}\big)\dot{\varGamma}$$

for all $\dot{X} \in \big\{\dot{F}_i, K^{\pm 1}, \dot{K}, \dot{\varGamma}, \dot{E}_i\big\}_{i=1,\dots,n}$ and $i, j = 1, \dots, n$, with Hopf structure given by

$$\begin{array}{cccc}
\Delta\big(\dot{E}_i\big) = \dot{E}_i \otimes K + 1 \otimes \dot{E}_i\,, & \epsilon\big(\dot{E}_i\big) = 0\,, & S\big(\dot{E}_i\big) = -\dot{E}_i K^{-1} & \forall\, i = 1, \dots, n \\
\Delta\big(K^{\pm 1}\big) = K^{\pm 1} \otimes K^{\pm 1}\,, & \epsilon\big(K^{\pm 1}\big) = 1\,, & S\big(K^{\pm 1}\big) = K^{\mp 1} & \\
\Delta\big(\dot{\varGamma}\big) = \dot{\varGamma} \otimes K + K^{-1} \otimes \dot{\varGamma}\,, & \epsilon\big(\dot{\varGamma}\big) = 0\,, & S\big(\dot{\varGamma}\big) = -\varGamma & \\
\Delta\big(\dot{H}\big) = \dot{H} \otimes 1 + K \otimes \dot{H}\,, & \epsilon\big(\dot{H}\big) = 0\,, & S\big(\dot{H}\big) = -K^{-1}\dot{H} & \\
\Delta\big(\dot{F}_i\big) = \dot{F}_i \otimes 1 + K^{-1} \otimes \dot{F}_i\,, & \epsilon\big(F_i\big) = 0\,, & S\big(\dot{F}_i\big) = -K\dot{F}_i & \forall\, i = 1, \dots, n\,.
\end{array}$$

As $q \to 1$, the presentation above provides an isomorphism of Poisson Hopf algebras

$$\widetilde{U}_q^s(\mathfrak{h}_n) \Big/ (q-1)\, \widetilde{U}_q^s(\mathfrak{h}_n) \xrightarrow{\ \cong\ } F\big[{}_sH_n{}^*\big]$$

given by $\dot{E}_i \mod (q-1) \mapsto \alpha_i \gamma^{+1}$, $L^{\pm 1} \mod (q-1) \mapsto \gamma^{\pm 1}$, $\dot{D} \mod (q-1) \mapsto \gamma - 1$, $\dot{\varGamma} \mod (q-1) \mapsto \big(\gamma^2 - \gamma^{-2}\big)\Big/2$, $\dot{F}_i \mod (q-1) \mapsto \gamma^{-1}\beta_i$. In other words, the semiclassical limit of $\widetilde{U}_q^s(\mathfrak{h}_n)$ is $F\big[{}_sH_n{}^*\big]$, as predicted by Theorem 2.1. Similarly, when considering the "adjoint case", we find a Poisson Hopf algebra isomorphism

$$\widetilde{U}_q^a(\mathfrak{h}_n) \Big/ (q-1)\, \widetilde{U}_q^a(\mathfrak{h}_n) \xrightarrow{\ \cong\ } F\big[{}_aH_n{}^*\big] \quad \Big(\subset F\big[{}_sH_n{}^*\big]\Big)$$

given by $\dot{E}_i \mod (q-1) \mapsto \alpha_i \gamma^{+1}$, $K^{\pm 1} \mod (q-1) \mapsto \gamma^{\pm 2}$, $\dot{H} \mod (q-1) \mapsto \gamma^2 - 1$, $\dot{\varGamma} \mod (q-1) \mapsto \big(\gamma^2 - \gamma^{-2}\big)\Big/2$, $\dot{F}_i \mod (q-1) \mapsto \gamma^{-1}\beta_i$. That is to say, $\widetilde{U}_q^a(\mathfrak{h}_n)$ has semiclassical limit $F\big[{}_aH_n{}^*\big]$, as predicted by Theorem 2.1.

5.4 The identity $\widetilde{\widetilde{U}}_q(\mathfrak{h}_n) = \widehat{U}_q(\mathfrak{h}_n)$. In this section we verify the first half of part *(b)* of Theorem 2.1 (*"The functors $\big(U_q, \widehat{U}_q\big) \longmapsto \big(U_q, \widetilde{U}_q\big)$ and $\big(F_q, \widehat{F}_q\big) \longmapsto \big(F_q, \widetilde{F}_q\big)$ are inverse of each other"*) for $\mathfrak{g} = \mathfrak{h}_n$. In other words, we check that $\widetilde{\widetilde{U}}_q^s(\mathfrak{h}_n) = \widehat{U}_q^s(\mathfrak{h}_n)$ and $\widetilde{\widetilde{U}}_q^a(\mathfrak{h}_n) = \widehat{U}_q^a(\mathfrak{h}_n)$. To begin with, using (5.1) and the fact that $\dot{F}_i, \dot{D}, \dot{\varGamma}, \dot{E}_i \in Ker\Big(\epsilon \colon \widetilde{U}_q^s(\mathfrak{h}_n) \twoheadrightarrow k\big[q, q^{-1}\big]\Big)$ we get that $J := Ker(\epsilon)$ is the $k\big[q, q^{-1}\big]$–span of $\mathbb{M} \setminus \{1\}$, where $\mathbb{M}$ is the set in the right-hand-side of (5.1). Since $\widetilde{\widetilde{U}}_q^s(\mathfrak{h}_n) := \sum_{n \geq 0} \Big((q-1)^{-1} I\Big)^n$ with $I := Ker\Big(\widetilde{U}_q^s(\mathfrak{h}_n) \xrightarrow{\ \epsilon\ } k\big[q, q^{-1}\big] \xrightarrow{\ ev_1\ } k\Big) = J + (q-1) \cdot \widetilde{U}_q^s(\mathfrak{h}_n)$ we have that $\widetilde{\widetilde{U}}_q^s(\mathfrak{h}_n)$ is generated — as a unital $k\big[q, q^{-1}\big]$–subalgebra of $U_q^s(\mathfrak{h}_n)$ — by $(q-1)^{-1} \dot{F}_i = F_i$, $(q-1)^{-1} \dot{D} = D$, $(q-1)^{-1} \dot{\varGamma} = \varGamma$, $(q-1)^{-1} \dot{E}_i = E_i$ $(i = 1, \dots, n)$, so it coincides with $\widehat{U}_q^s(\mathfrak{h}_n)$, q.e.d. The situation is entirely similar for the adjoint case: one simply has to change $L^{\pm 1}$, resp. $\dot{D}$, with $K^{\pm 1}$, resp. $\dot{H}$, then everything goes through as above.

5.5 The QFA $\big(F_q[H_n], \widehat{F}_q[H_n]\big)$. We now look at Theorem 2.1 the other way round, i.e. from QFA's to (G)QUEA's. We begin by introducing a QFA for the Heisenberg group.

Let $\widehat{F}_q[H_n]$ be the unital associative $k\big[q, q^{-1}\big]$–algebra with generators $\mathrm{a}_1, \dots, \mathrm{a}_n$, c, $\mathrm{b}_1, \dots, \mathrm{b}_n$, and relations (for all $i, j = 1, \dots, n$)

$$\mathrm{a}_i \mathrm{a}_j = \mathrm{a}_j \mathrm{a}_i\,, \ \ \mathrm{a}_i \mathrm{b}_j = \mathrm{b}_j \mathrm{a}_i\,, \ \ \mathrm{b}_i \mathrm{b}_j = \mathrm{b}_j \mathrm{b}_i\,, \ \ \mathrm{a}_i \mathrm{c} = \mathrm{c}\mathrm{a}_i + (q-1)\mathrm{a}_i\,, \ \ \mathrm{b}_j \mathrm{c} = \mathrm{c}\mathrm{b}_j + (q-1)\mathrm{b}_j$$

with a Hopf algebra structure given by (for all $i, j = 1, \dots, n$)

$$\Delta(\mathrm{a}_i) = \mathrm{a}_i \otimes 1 + 1 \otimes \mathrm{a}_i\,, \quad \Delta(\mathrm{c}) = \mathrm{c} \otimes 1 + 1 \otimes \mathrm{c} + \sum_{j=1}^n \mathrm{a}_\ell \otimes \mathrm{b}_\ell\,, \quad \Delta(\mathrm{b}_i) = \mathrm{b}_i \otimes 1 + 1 \otimes \mathrm{b}_i$$

$$\epsilon(\mathrm{a}_i) = 0\,, \ \ \epsilon(\mathrm{c}) = 0\,, \ \ \epsilon(\mathrm{b}_i) = 0\,, \qquad S(\mathrm{a}_i) = -\mathrm{a}_i\,, \ \ S(\mathrm{c}) = -\mathrm{c} + \sum_{j=1}^n \mathrm{a}_\ell \mathrm{b}_\ell\,, \ \ S(\mathrm{b}_i) = -\mathrm{b}_i$$

and let also $F_q[H_n]$ be the $k(q)$–algebra obtained from $\widehat{F}_q[H_n]$ by scalar extension. Then $\mathbb{B} := \Big\{ \prod_{i=1}^n \mathrm{a}_i^{a_i} \cdot \mathrm{c}^c \cdot \prod_{j=1}^n \mathrm{b}_j^{b_j} \,\Big|\, a_i, c, b_j \in \mathbb{N}\ \forall\, i, j \Big\}$ is a $k\big[q,q^{-1}\big]$–basis of $\widehat{F}_q[H_n]$, hence a $k(q)$–basis of $F_q[H_n]$. Moreover, $\big(F_q[H_n], \widehat{F}_q[H_n]\big)$ is a QFA, with semiclassical limit $F[H_n]$.

5.6 Computation of $\widetilde{F}_q[H_n]$ and specialization $\widetilde{F}_q[H_n] \xrightarrow{\,q\to 1\,} U(\mathfrak{h}_n{}^*)$. This section is devoted to compute $\widetilde{F}_q[H_n]$ and its semiclassical limit $\widetilde{F}_1[H_n]$.

Definitions imply that $\mathbb{B}\setminus\{1\}$ is a $k\big[q,q^{-1}\big]$–basis of $J := Ker\Big(\epsilon \colon \widehat{F}_q\big[H_n\big] \twoheadrightarrow k\big[q,q^{-1}\big]\Big)$, so $\big(\mathbb{B}\setminus\{1\}\big) \cup \big\{(q-1)\cdot 1\big\}$ is a $k\big[q,q^{-1}\big]$–basis of $I := Ker\Big(\widehat{F}_q\big[H_n\big] \stackrel{\epsilon}{\twoheadrightarrow} k\big[q,q^{-1}\big] \stackrel{ev_1}{\twoheadrightarrow} k\Big)$, for $I = J + (q-1)\cdot \widehat{F}_q\big[H_n\big]$. Therefore $\widetilde{F}_q\big[H_n\big] := \sum_{n \geq 0} \Big((q-1)^{-1} I\Big)^n$ is nothing but the unital $k\big[q,q^{-1}\big]$–algebra (subalgebra of $F_q\big[H_n\big]$) with generators $E_i := \dfrac{\mathrm{a}_i}{q-1}$, $H := \dfrac{\mathrm{c}}{q-1}$, and $F_i := \dfrac{\mathrm{b}_i}{q-1}$ ($i = 1, \dots, n$) and relations (for all $i, j = 1, \dots, n$)

$$E_i E_j = E_j E_i\,, \quad E_i F_j = F_j E_i\,, \quad F_i F_j = F_j F_i\,, \quad E_i H = H E_i + E_i\,, \quad F_j H = H F_j + F_j$$

with Hopf algebra structure given by (for all $i, j = 1, \dots, n$)

$$\Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i\,, \;\; \Delta(H) = H \otimes 1 + 1 \otimes H + (q-1) \sum_{j=1}^n E_j \otimes F_j\,, \;\; \Delta(F_i) = F_i \otimes 1 + 1 \otimes F_i$$

$$\epsilon(E_i) = 0,\ \epsilon(F_i) = 0,\ \epsilon(H) = 0,\ \ S(E_i) = -E_i,\ S(H) = -H + (q-1) \sum_{j=1}^n E_j F_j,\ S(F_i) = -F_i.$$

At $q=1$ this implies that $\widetilde{F}_q\big[H_n\big] \xrightarrow{\,q\to 1\,} U(\mathfrak{h}_n{}^*)$ as co-Poisson Hopf algebras, for a co-Poisson Hopf algebra isomorphism

$$\widetilde{F}_1\big[H_n\big] := \widetilde{F}_q\big[H_n\big] \Big/ (q-1)\, \widetilde{F}_q\big[H_n\big] \xrightarrow{\;\cong\;} U(\mathfrak{h}_n{}^*)$$

exists, given by $E_i \mod (q-1) \mapsto \pm \mathrm{e}_i$, $H \mod (q-1) \mapsto \mathrm{h}$, $F_i \mod (q-1) \mapsto \mathrm{f}_i$, for all $i, j = 1, \dots, n$; thus $\widetilde{F}_q\big[H_n\big]$ specializes to $U(\mathfrak{h}_n{}^*)$ *as a co-Poisson Hopf algebra*, q.e.d.

5.7 The identity $\widehat{\widetilde{F}}_q[H_n] = \widehat{F}_q[H_n]$. Finally, we check the validity of the second half of part *(b)* of Theorem 2.1 (*"The functors $\big(U_q, \widehat{U}_q\big) \mapsto \big(U_q, \widetilde{U}_q\big)$ and $\big(F_q, \widehat{F}_q\big) \mapsto \big(F_q, \widetilde{F}_q\big)$ are inverse of each other"*), i.e. that $\widehat{\widetilde{F}}_q[H_n] = \widehat{F}_q[H_n]$. First, induction gives, for all $m \in \mathbb{N}$,

$$\Delta^m(E_i) = \sum_{r+s=m-1} 1^{\otimes r} \otimes E_i \otimes 1^{\otimes s}\,, \qquad \Delta^m(F_i) = \sum_{r+s=m-1} 1^{\otimes r} \otimes F_i \otimes 1^{\otimes s} \qquad \forall\, i = 1, \dots, n$$

$$\Delta^m(H) = \sum_{r+s=m-1} 1^{\otimes r} \otimes H \otimes 1^{\otimes s} + \sum_{i=1}^m \sum_{\substack{j,k=1 \\ j<k}}^m 1^{\otimes (j-1)} \otimes E_i \otimes 1^{\otimes (k-j-1)} \otimes F_i \otimes 1^{\otimes (m-k)}$$

so that $\delta_m(E_i) = \delta_\ell(H) = \delta_m(F_i) = 0$ for all $m > 1$, $\ell > 2$ and $i = 1, \dots, n$; moreover,

for $\dot{E}_i := (q-1)E_i = \mathrm{a}_i$, $\dot{H} := (q-1)H = \mathrm{c}$, $\dot{F}_i := (q-1)F_i = \mathrm{b}_i$ $(i = 1, \dots, n)$ one has $\delta_1\big(\dot{E}_i\big) = (q-1)E_i$, $\delta_1\big(\dot{H}\big) = (q-1)H$, $\delta_1\big(\dot{F}_i\big) = (q-1)F_i \in (q-1)\,\widetilde{F}_q\big[H_n\big] \setminus (q-1)^2 \widetilde{F}_q\big[H_n\big]$

$$\delta_2\big(\dot{H}\big) = (q-1)^2 \sum_{i=1}^n E_i \otimes F_i \in (q-1)^2 \widetilde{F}_q\big[H_n\big]^{\otimes 2} \setminus (q-1)^3 \widetilde{F}_q\big[H_n\big]^{\otimes 2} .$$

The upset is that $\dot{E}_i = \mathrm{a}_i, \dot{H} = \mathrm{c}, \dot{F}_i = \mathrm{b}_i \in \widetilde{\widetilde{F}}_q\big[H_n\big]$, so the latter algebra contains the one generated by these elements, that is $\widehat{F}_q\big[H_n\big]$. Even more, it is clear that $\widetilde{F}_q\big[H_n\big]$ is the $k\big[q,q^{-1}\big]$–span of the set $\widetilde{\mathbb{B}} := \Big\{ \prod_{i=1}^n E_i^{a_i} \cdot H^c \cdot \prod_{j=1}^n F_j^{b_j} \,\Big|\, a_i, c, b_j \in \mathbb{N} \;\forall\, i,j \Big\}$, so from this and the previous formulae for Δ^n one gets that $\widetilde{\widetilde{F}}_q\big[E_2\big]$ coincides with $\widehat{F}_q\big[E_2\big]$, q.e.d.

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